On the associated martingale for a multitype branching process in random environment

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What is the talk about

1. With a branching process in random environment (with one type on particles) (Z_n) one can associate a martingale which is used to show that (under assumptions) the size of the population exploses:
$$Z_n \asymp m_1 \dots m_n.$$
where m_k are the quenched reproduction means. For fixed deterministic environment (:= no environment) this simply reads
$$Z_n \asymp m^n.$$
2. A similar result holds for a multitype branching process (without environment = fixed deterministic environment). This is the famous Kesten-Stigum theorem.
3. However, until recently there was no completely satisfactory analog of this property in the case of a multitype branching process in random environment. Previous results: Cohn (1989), Jones (1997) [L^2-convergence of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$], Biggins, Cohn, Nerman (1999) [in L^p], Le Page, Peigné, Pham (2019).

Our contribution

- The main difficulty is the construction of the associated martingale, which is the main tool in establishing the K-S theorem.
- Our goal is to complement on the construction of this martingale in G.-Liu-Pin, AAP 2023, by considering a triangular array of martingales and by showing the convergence of its terminal values.
- Usefulness: this construction is used to prove the Berry-Esseen theorem, to establish Moderate deviations, and with the last developments also a precise Large deviation asymptotic (in progress).
- The construction of the associated martingale is related to a "new" version of the Perron-Frobenius theorem for products of random matrices.

Outline

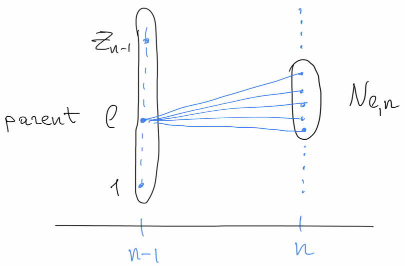
1. Start with the case of 1 type of particles.
2. Then we will pass to multitype case: Kesten-Stigum theorem.
3. We will state a Perron-Frobenius theorem for products of random matrices, construct the martingale and state an analog of the K-S theorem.

Single-type BP

Consider a single type branching process in random environment:

$$Z_0 = 1, \quad Z_n = \sum_{l=1}^{Z_{n-1}} N_{l,n}, \quad n = 1, 2, \ldots$$

- $N_{l,n}$ is the number of children generated by the parent l in generation $n-1$
  $N_{1,n}, N_{2,n}, \ldots$ are i.i.d. with p.g.f. $f_n(s) = f_{\xi_n}(s)$.
- The environment sequence $\xi = (\xi_0, \xi_1, \ldots)$ is i.i.d.



The martingale for single-type BP

1. The reproduction mean in generation n is denoted by

$$m_n = m(\xi_n) = \mathbb{E}_{\xi_n} N_{l,n} = \frac{\partial}{\partial s} f_{\xi_n}(1).$$

This is a sequence of i.i.d. random variables depending only on ξ.

2. The following process is a martingale

$$W_0 = 1, \quad W_n = \frac{Z_n}{m_1 \dots m_n}, \quad n \geqslant 1. \qquad \left(W_n = \frac{Z_n}{m^n} \right)$$

with respect to the quenched measure $\mathbb{P}_\xi$ and the filtration

$$\mathscr{F}_n = \sigma\{\xi, N_{l,k},\ k \leqslant n,\ \forall l\},$$

Proof: Use the simple fact that $\mathbb{E}(Z_n | \mathscr{F}_{n-1}) = Z_{n-1} m_n$.

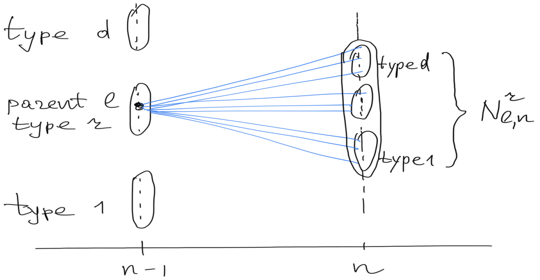
Multitype branching process

Consider a branching process with d types of particles (no environment):

$$Z_n = (Z_n(1), \dots, Z_n(d)), \quad Z_n = \sum_{r=1}^{d} \sum_{l=1}^{Z_{n-1}} N_{l,n}^r, \quad n = 1, 2, \dots,$$

- $N_{l,n}^r$ is the row-vector of children (of all types) generated by the parent l of type r in generation $n-1$:
- the sequence $N_{1,n}^r, N_{2,n}^r, \dots$ is i.i.d. and independent of the past

$$\mathscr{F}_{n-1} = \sigma\{N_{1,n-1}^r, N_{2,n-1}^r, \dots\}.$$



Matrix of the means

1. With a constant deterministic environment, the mean number of born children is a (constant non-random) matrix M, whose entries

$$M(r,j) = \mathbb{E}\, N^r_{l,n}(j)$$

are the mean production of children of type j by any parent of type r.

2. In an i.i.d. random environment $\xi = (\xi_0, \xi_1, \ldots)$ we will have matrices (M_n) changing with n:

$$M_n(r,j) = \mathbb{E}\left(N^r_{l,n}(j)|\xi\right) = \mathbb{E}\left(N^r_{l,n}(j)|\xi_n\right)$$

each depending on the environment variable ξ_n.

- Since the sequence (ξ_n) is i.i.d. it follows that the sequence of matrices (M_n) is also i.i.d.

Kesten-Stigum theorem

Consider a MBP (no environment). The (non-random) mean matrix M is assumed to be primitive ($M^k > 0$ for some $k \geqslant 1$).

- Let ρ be the spectral radius of M which is dominating eigenvalue of multiplicity 1.
- By the Perron-Frobenius theorem, there exist unique $u > 0$ and $v > 0$ which are the right and left row-eigenvectors of M, that is

$$Mu^T = \rho u^T, \quad vM = \rho v, \quad \text{with} \quad \|u\| = 1, \ \langle v, u \rangle = 1.$$

Theorem (Kesten-Stigum 1966)

1. Part 1: for any $1 \leqslant i, j \leqslant d$ it holds, with some r.v. $W^i \geqslant 0$,

$$\frac{Z_n^i(j)}{\rho^n u(i) v(j)} \to W^i \quad \mathbb{P}\text{-a.s. as } n \to \infty. \tag{1}$$

2. Part 2: the limits W^i are non degenerate for all $1 \leqslant i \leqslant d \Leftrightarrow$ $\mathbb{E} Z_1^i(j) \log^+ Z_1^i(j) < \infty$, for all $1 \leqslant i, j \leqslant d$.

Notation: Z_n^i means that the BP starts with 1 particle of type i.

Equivalent formulation

1. In addidtion to the previous the Perron-Frobenius theorem tells that $\lim_{n\to\infty} \frac{1}{\rho^n} M^n = u \otimes v$; in the component form becomes:

$$M^n(i,j) \sim \rho^n u(i)v(j), \quad \text{for any } 1 \leqslant i,j \leqslant d.$$

2. Then Part 1 of the K-S theorem (on previous slide) is equivalent to:

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M^n(i,j)} \to W^i \quad \mathbb{P}\text{-a.s. as } n \to \infty. \tag{2}$$

The relation (2) is an analog of the convergence stated for the BP with 1 type of particles. It can be rewritten (with $x = e_i,\ y = e_j$):

$$\frac{\langle Z_n^x, y\rangle}{\langle x\, M^n, y\rangle} \to W^x. \tag{3}$$

3. Note that $\frac{Z_n^i(j)}{M^n(i,j)}$, $n \geqslant 0$ is not a martingale as in the case $d = 1$.

The associated martingale

1. The K-S theorem is based on the following martingale: for $n \geqslant 0$,

$$W_n = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)} = \frac{\langle Z_n^{e_i}, u \rangle}{\langle e_i M^n, u \rangle}, \quad n \geqslant 0,$$

which converges $\mathbb{P}_\xi$-a.s. to W^i.

(Recall: u is the right eigenvector of M: $Mu^T = \rho u^T$).

2. Proof. We use the simple property: $E_\xi \left(Z_n | \mathscr{F}_{n-1} \right) = Z_{n-1} M$. Thus

$$\begin{aligned}
\mathbb{E}_\xi \left(W_n | \mathscr{F}_{n-1} \right) &= \frac{\langle \mathbb{E}_\xi \left(Z_n^{e_i} | \mathscr{F}_{n-1} \right), u \rangle}{\langle e_i M^n, u \rangle} \\
&= \frac{\langle Z_{n-1}^{e_i} M, u \rangle}{\langle e_i M^n, u \rangle} = \frac{\langle Z_{n-1}^{e_i}, uM^T \rangle}{\langle e_i M^{n-1}, uM^T \rangle} = \frac{\rho \langle Z_{n-1}^{e_i}, u \rangle}{\rho \langle e_i M^{n-1}, u \rangle} \\
&= \frac{\langle Z_{n-1}^{e_i}, u \rangle}{\langle e_i M^{n-1}, u \rangle} = W_{n-1}.
\end{aligned}$$

3. Recall: until recently there was no extension to the case of a multitype BP in random environment. Why ?

Martingale extension: naive attempt

1. Recall that with a random environment, we have a sequence of i.i.d. matrices (M_n).
2. By analogy with the K-S construction set: for any x, y

$$W_n^x(y) = \frac{\langle Z_n^x, y\rangle}{\langle xM_1 \dots M_n, y\rangle}, \quad n \geqslant 0.$$

Question: what we should choose for y ?

3. Let $y = y_n$ where y_n is the right eigenvector of the matrix M_n: $y_n M_n^T = \rho_n y_n$. Then, using $E_\xi\left(Z_n^x | \mathscr{F}_{n-1}\right) = Z_{n-1}^x M_n$,

$$\mathbb{E}_\xi\left(W_n^x(y_n) | \mathscr{F}_{n-1}\right) = \frac{\langle E_\xi\left(Z_n^x | \mathscr{F}_{n-1}\right), y_n\rangle}{\langle xM_1 \dots M_n, y_n\rangle}$$

$$= \frac{\langle Z_{n-1}^x M_n, y_n\rangle}{\langle xM_1 \dots M_n, y_n\rangle} = \frac{\langle Z_{n-1}^x, y_n M_n^T\rangle}{\langle xM_1 \dots M_{n-1}, y_n M_n^T\rangle}$$

$$= \frac{\rho_n \langle Z_{n-1}^x, y_n\rangle}{\rho_n \langle xM_1 \dots M_{n-1}, y_n\rangle} = \frac{\langle Z_{n-1}^x, y_n\rangle}{\langle xM_1 \dots M_{n-1}, y_n\rangle} \neq W_{n-1}.$$

To get a martingale we need the property $y_n M_n^T = \lambda_n y_{n-1}$.

Dolgopyat, Hebbar, Koralov, Perlman (2018). [Seneta (1981)]

Recall the Perron-Frobenius theorem

Theorem

Assume that the matrix M has positive entries. Denote by $\rho = \rho(M)$ its spectral radius. Then

1. $\rho > 0$ and is an eigenvalue of the matrix M. Any other (possibly complex) eigenvalue in absolute value is strictly smaller than ρ. The eigenvalue ρ is simple and right and left eigenspaces associated with ρ are one-dimensional.
2. There exists a right eigenvector $\mathbf{u} > 0$ such that $\mathbf{M}\mathbf{u}^{\mathbf{T}} = \rho\mathbf{u}^{\mathbf{T}}$.

   There exists a left eigenvector $\mathbf{v} > 0$ such that $\mathbf{vM} = \rho\mathbf{v}$.

   The vectors $\mathbf{u}$ and $\mathbf{v}$ can be chosen uniquely in such a way that

   $\|\mathbf{u}\| = \mathbf{1}$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{1}$.
3. In addition, it holds $\lim_{n\to\infty} \frac{1}{\rho^n} M^n = \mathbf{u} \otimes \mathbf{v}$, where the matrix $\mathbf{u} \otimes \mathbf{v}$ is the projection onto the subspace generated by $\mathbf{u}$.

These statements extend to a primitive M, i.e. $M \geqslant 0$ and $M^k > 0$ for some $k \geqslant 1$.

Perron-Frobenius theorem

- The point 3 of the previous theorem, i.e.

$$\lim_{n\to\infty} \frac{1}{\rho^n} M^n = \mathbf{u} \otimes \mathbf{v},$$

can we rewritten in the following equivalent way:

- for any $1 \leqslant i, j \leqslant d$,

$$\lim_{n\to\infty} \frac{\langle e_i M^n, e_j \rangle}{\rho^n \langle u, e_i \rangle \langle v, e_j \rangle} = 1,$$

- or, for any $x, y \in \mathbb{R}^d$, $x, y \neq 0$ (instead of e_i, e_j),

$$\lim_{n\to\infty} \frac{\langle x M^n, y \rangle}{\rho^n \langle u, x \rangle \langle v, y \rangle} = 1.$$

A Perron-Frobenius theorem for random matrices

Consider the i.i.d. random matrices M_k indexed with $k \in \mathbb{Z}$.

Assume condition A1:

1. The matrices M_k satisfy $M_k \geqslant 0$ and are allowable (every row and every column contains a strictly positive entry).

2. The Hennion condition: $\mathbb{P}(\exists k \text{ such that } M_1 \dots M_k > 0) = 1$.

   This is an analog of the condition "$M^k > 0$ for some $k \geqslant 0$" ("M is primitive").

A Perron-Frobenius theorem for (M_n)

Theorem 1.

Assume A1 (allowability + Hennion condition):

1. There exists a stationary and ergodic sequence of vectors $u_n > 0$, $\|u_n\| = 1$, $n \in \mathbb{Z}$:
$$u_{n+1}\, M_n^T = \lambda_n u_n, \qquad \lambda_n = \|M_n u_{n+1}\|.$$
2. There exists a stationary and ergodic sequence of vectors $v_n > 0$, $\|v_n\| = 1$, $n \in \mathbb{Z}$:
$$v_{n-1}\, M_n = \mu_n v_n, \qquad \mu_n = \|v_{n-1} M_n\|.$$
3. For any vectors x and y,
$$\lim_{n\to\infty} \frac{\langle x\, M_k \dots M_n, y\rangle}{d_{k,n}\langle u_k, x\rangle\langle v_n, y\rangle} = 1,$$
where $d_{k,n} := \langle 1, 1 M_k \dots M_n\rangle = \sum_{i,j} M(i,j)$.

Relation to eigenvectors

1. Let $\rho_{k,n}$, $u_{k,n}$ and $v_{k,n}$ be the spectral radius, the right and the left eigenvectors of the matrix $M_k \dots M_n$, i.e.

$$u_{k,n}(M_k \dots M_n)^T = \rho_{k,n} u_{k,n} \qquad v_{k,n} M_k \dots M_n = \rho_{k,n} v_{k,n}.$$

with constraints $\|u_{k,n}\| = 1$ and $\langle u_{k,n}, v_{k,n} \rangle = 1$.
We have a.s.

$$\lim_{n\to\infty} u_{k,n} = u_k, \qquad \lim_{k\to-\infty} \frac{v_{k,n}}{\|v_{k,n}\|} = v_n. \tag{4}$$

2. Comparison with Hennion (1997) result: as $n \to \infty$ the convergence for v_n holds only in law: for fixed k

$$\overline{v}_{k,n} := \frac{v_{k,n}}{\|v_{k,n}\|} \overset{d}{\to} v_k \qquad \Leftarrow \left(\frac{\langle \overline{v}_{k,n}, y \rangle}{\langle v_n, y \rangle} \to 1 \text{ a.s. unif.} \forall y \neq 0. \right)$$

Associated martingale for MBPRE

1. Using the sequence $y_n = u_{n+1}$, where $u_n > 0$, $n \in \mathbb{Z}$ is stationary and ergodic and satisfies $M_n\, u_{n+1}^T = \lambda_n u_n^T$ and $\|u_n\| = 1$, we obtain

Theorem 2

Under A1 (allowability + Hennion condition), the sequence

$$W_n^x(u_{n+1}) = \frac{\langle Z_n^x, u_{n+1}\rangle}{\langle x\, M_1 \dots M_n, u_{n+1}\rangle}$$

is a positive martingale.

2. By the martingale convergence theorem there exist the following limit

$$W_n^x(u_{n+1}) = \frac{\langle Z_n^x, u_{n+1}\rangle}{\langle xM_1 \dots M_n, u_{n+1}\rangle} \to W^x,$$

where $W^x \geqslant 0$. We shall discuss its non-degeneracy below.

3. We still need to show a relation between $W_n^x(u_{n+1})$ and the quantity we are interested in $W_n^x(y) = \frac{\langle Z_n^x, y\rangle}{\langle xM_1 \dots M_n, y\rangle}$.

A triangular array of martingales

1. Again using the property $E_\xi\left(Z_n \middle| \mathscr{F}_{n-1}\right) = Z_{n-1} M_n$ we can easily check that, for any $n \geqslant 0$ and any x, y,

$$W^x_{n,k}(y) = \frac{\langle Z^x_k M_{k+1} \dots M_n, y \rangle}{\langle x\, M_1 \dots M_n, y \rangle}, \quad k = 0, \dots, n$$

is a triangular array of finite time $\mathbb{P}_\xi$-martingales.

$W^x_{00}(y)$

$W^x_{10}(y) \quad W^x_{11}(y)$

$W^x_{20}(y) \quad W^x_{21}(y) \quad W^x_{22}(y)$

...

$W^x_{n0}(y) \quad W_{n1}(y) \quad \dots \quad W^x_{nn}(y)$

2. Its terminal values are exactly the quantities of interest:

$$W^x_{n,n}(y) = W^x_n(y) = \frac{\langle Z^x_n, y \rangle}{\langle x M_1 \dots M_n, y \rangle}.$$

Kesten-Stigum theorem

Theorem 3:

1. Assume: A1 (allowability + Hennion condition),
   A2 ($\mathbb{E}\log^+ \|M_0\| < +\infty$).

   Then
   $$\lim_{n\to\infty} \frac{W_n^x(y)}{W_n^x(u_{n+1})} = 1, \quad \text{in probability } \mathbb{P}, \quad \forall x, y. \tag{5}$$
   conditioned on the explosion event $E^x = \{\lim_{n\to\infty} Z_n^x = \infty\}$.

2. As a consequence, for any x and y, as $n \to \infty$,
   $$W_n^x(y) = \frac{\langle Z_n^x, y\rangle}{\langle x\, M_1 \ldots M_n, y\rangle} \to W^x, \quad \text{in probability } \mathbb{P}, \tag{6}$$
   where W^x is the limit of the martingale $(W_n^x(u_{n+1}))_{n\geqslant 0}$.

   This is the analog of the Part 1 of the Kesten-Stigum theorem (convergence to a limit).

The convergence is in probability only (since we have a triangular array of martingales).
For the a.s. convergence we need some additional conditions.

K-S theorem: a.s. convergence

- Assume additionally that for some $p > 1$ and for all $1 \leqslant r \leqslant d$,

$$\mathbb{E} \sup_{y \in \mathbb{R}_+^d \setminus \{0\}} \left(\frac{\langle Z_1^r, y \rangle}{\langle e_r M_1, y \rangle} \right)^p < +\infty \tag{7}$$

and

$$\mathbb{E} \| M_1 \|^{1-p} < +\infty, \tag{8}$$

Then , for any x and y, as $n \to \infty$, the convergence in the above theorem is $\mathbb{P}$-a.s.

Non-degeneracy for supercritical MBPRE's

1. We prove the non-degeneracy of W^x for a supercritical MBPRE.
   What is definition of the supercriticality ?
2. The following strong law of large numbers has been established by Furstenberg and Kesten 1960: under A2 ($\mathbb{E} \log^+ \|M_1\| < +\infty$),

$$\lim_{n \to +\infty} \frac{1}{n} \log \|M_1 \dots M_n\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

3. The Lyapunov exponent γ allows to introduce the following classification of MBPRE's:

Definition

We say that $(Z_n)_{n \geqslant 0}$ is:
subcritical if $\gamma < 0$; critical if $\gamma = 0$; supercritical if $\gamma > 0$.

This def. sticks with the definition for a single-type BP.

Non-degeneracy of W^x

- In the following we consider supercritical MBPRE's: $\gamma > 0$. We give a sufficient condition for the non-degeneracy of W^x.
- Condition H2: For all $1 \leqslant r \leqslant d$,

$$\mathbb{E}\left(\frac{\langle N_{1,1}^r, u_1\rangle}{\lambda_1 \langle u_1, e_r\rangle} \log^+ \langle N_{1,1}^r, u_1\rangle\right) < +\infty. \tag{9}$$

Theorem 3:

Assume: A1(allowability + Hennion condition),
A2 ($\mathbb{E} \log^+ \|M_1\| < +\infty$), $\gamma > 0$ (supercritical).

1. Then H2 is a sufficient condition for W^x to be non-degenerate $\forall x$.
2. Furthermore, when W^x, for $\forall x \neq 0$ are non-degenerate, we have $\mathbb{E}_\xi W^x = 1$ for $\forall x \neq 0$, $\mathbb{P}$-a.s.

Necessary and sufficient condition

We need stronger conditions:

- (F-K) The Furstenberg-Kesten condition: $\frac{max_{i,j} M_1(i,j)}{min_{i,j} M_1(i,j)} \leqslant C$
- Condition H3: For all $1 \leqslant r \leqslant d$, For all $1 \leqslant r, j \leqslant d$,

$$\mathbb{E}\left[\frac{N^r_{1,1}(j)}{\langle e_r M_1, e_j\rangle} \log^+ \frac{N^r_{1,1}(j)}{\langle e_r M_1, e_j\rangle}\right] < +\infty.$$

Theorem 5:

Assume: F-K, A2 ($\mathbb{E} \log^+ \|M_1\| < +\infty$), $\gamma > 0$ (supercritical).

1. Then H3 is a necessary and sufficient condition for W^x to be non-degenerate $\forall x$.
2. Furthermore, when W^x, for $\forall x \neq 0$ are non-degenerate, we have $\mathbb{E}_\xi W^x = 1$ for $\forall x \neq 0$, $\mathbb{P}$-a.s.

Proof: we use the method based on size biased tree by Lyons, Permantle and Peres (1995) [Bigging and Kyprianou (2004)].

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Thank you !!!