Duality, intertwining and orthogonal polynomials for interacting particles in $\mathbb{R}^d$

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Duality of Markov processes with respect to a function:
Useful tool in queuing theory, random walks with reflecting / absorbing barriers, mathematical population genetics, interacting particle systems ...

Basic idea:
In order to understand long-time behavior of complicated process, follow simpler process backwards in time.

In interacting particle systems:
often duality allows to map study of k-point correlation function of many-particle system to time evolution of a system with k particles.
Holley, Stroock '79; Liggett's book on interacting particle systems.

Duality vs. intertwining

Process – state space – semigroup

$$(X_t)_{t\geq 0},\ E,\ (P_t)_{t\geq 0}, \qquad (Y_t)_{t\geq 0},\ F,\ (Q_t)_{t\geq 0}.$$

Dual with duality function $H : E \times F \to \mathbb{R}$:

$$\mathbb{E}_x\big[H(X_t, y)\big] = \mathbb{E}^y\big[H(x, Y_t)\big]$$

for all x, y, t.

Lévy '48: Brownian motion on half-line absorbed / reflected at 0, $H(x, y) = \mathbb{1}_{\{x \leq y\}}$.

Kernel $K(x, \mathrm{d}y)$ is an intertwiner if

$$\int P_t(x, \mathrm{d}x')K(x', B) = \int K(x, \mathrm{d}y)Q_t(y, B) \quad (B \subset F).$$

Duality & reversible measure $\mu \to$ intertwiner

$$K(x, \mathrm{d}y) = H(x, y)\mu(\mathrm{d}y).$$

The question

Franceschini, Giardinà 2018; Redig, Sau 2018

Several interacting particle systems on $\mathbb{Z}^d$ have factorized reversible measures ν on $\mathbb{N}_0^{\mathbb{Z}^d}$,

$$\nu = \otimes_{x \in \mathbb{Z}^d} \nu_x$$

ν_x = measure on $\mathbb{N}_0$, in shift-invariant setting: $\nu_x \equiv \nu_0$.

Often self-duality functions of product form

$$D(\boldsymbol{n}, \boldsymbol{m}) = \prod_{x \in \mathbb{Z}^d} D_0(n_x, m_x)$$

with single-site function

$$D_0(n, m) = \frac{P_n(m)}{\nu_0(m)},$$

$(P_n)_{n \in \mathbb{N}}$ discrete orthogonal polynomials, orthogonality w.r.t. ν_0.

What happens for $\mathbb{R}^d$ instead of $\mathbb{Z}^d$?

Overview

1. Consistent processes
2. Intertwining with Lenard's K-transform
3. Intertwining with orthogonal polynomials
4. Example: Symmetric inclusion process & Meixner polynomials

Consistency: labelled particles

Markov dynamics for finitely many particles in $E = \mathbb{R}^d$.

Total number of particles conserved.

Markov kernel for random removal of a point:

$$\Pi_{n+1,n}(x_1, \dots, x_{n+1}; B) = \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbb{1}_B(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}), \quad B \subset E^n.$$

(Weak) consistency of transition functions $p_t^{(n)}(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})$ on E^n:

$$p_t^{(n+1)} \Pi_{n+1,n} = \Pi_{n+1,n} p_t^{(n)}.$$

For $n = 1$

$$p_t^{(2)}(x_1, x_2; B \times E) + p_t^{(2)}(x_1, x_2; E \times B) = p_t^{(1)}(x_1; B) + p_t^{(1)}(x_2; B).$$

Example: independent random walkers.

Interacting diffusions, sticky Brownian motion: Le Jan, Raimond '04...

Howitt, Warren '09 *Consistent families of Brownian motion and stochastic flows of kernels*.

Consistency: non-labelled particles

Configuration $\{x_j\}_{j=1,\dots,n} \leftrightarrow$ counting measure $\eta = \delta_{x_1} + \cdots + \delta_{x_n}$

$$\eta(B) = \#\{j :\ x_j \in B\}.$$

$\mathbf{N}_\mathrm{f}$ space of finite counting measures.

Annihilation operator

$$\mathcal{A}f\left(\delta_{x_1} + \cdots + \delta_{x_n}\right) = \sum_{i=1}^{n} f\left(\delta_{x_1} + \cdots + \not{\delta_{x_i}} + \cdots + \delta_{x_n}\right)$$

Semigroup (P_t) on $\mathbf{N}_\mathrm{f}$ consistent if

$$P_t \mathcal{A} = \mathcal{A} P_t.$$

Independent random walkers / free Kawasaki

$$Lf(\eta) = \iint \big(f(\eta - \delta_x + \delta_y) - f(\eta)\big) q(x, \mathrm{d}y) \eta(\mathrm{d}x).$$

$q(x, \mathrm{d}y)$ one-particle jump kernel.

Symmetric inclusion process

$\alpha(\mathrm{d}\boldsymbol{x})$ finite measure on $\mathbb{R}^d$

Generator

$$Lf(\eta) = \iint \big(f(\eta - \delta_x + \delta_y) - f(\eta)\big)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x)$$

Particles jump.

Join another particle or jump to new location.

Symmetric inclusion process is consistent.

Variant: add spatial structure: $(\alpha + \eta)(\mathrm{d}y) \to c(x, y)(\alpha + \eta)(\mathrm{d}y)$, $c(x, y) = c(y, x)$.

Lattice version introduced as **dual model in energy transport** Giardinà, Kurchan, Redig 2007, reminiscent of Kipnis, Marchioro, Presutti 1982.

Intertwining with Lenard's K-transform

K-transform of function $f : \mathbf{N}_\mathrm{f} \to \mathbb{R}$

$$Kf\big(\delta_{x_1} + \cdots + \delta_{x_n}\big) = \sum_{I \subset \{1,\ldots,n\}} f\Big(\sum_{i\in I} \delta_{x_i}\Big).$$

Lenard' 73: *Correlation functions and the uniqueness of the state in classical mechanics.*

Theorem

(P_t) conservative semigroup on $\mathbf{N}_\mathrm{f}$. Then (P_t) is consistent if and only if

$$\forall t \geq 0: \quad P_t K = K P_t.$$

Similar relation for free Kawasaki dynamics Kondratiev, Kuna, Oliveira, da Silva, Streit 2009.

Lattice result Carinci, Giardinà, Redig 2019.

Consequence: for $\eta_0 = \delta_{x_1} + \cdots + \delta_{x_n}$, get

$$\mathbb{E}_{\eta_0}\big[\eta_t(B)\big] = \sum_{i=1}^n \mathbb{P}_{x_i}(X_t^{(1)} \in B).$$

Time evolution of k-point correlation functions $\leftrightarrow$ time-evolution for k particles.

Intertwining with orthogonal polynomials

ρ probability measure on $\mathbf{N}_\mathrm{f}$.

$\overline{\mathscr{P}_n}$ = closure in $L^2(\mathbf{N}_\mathrm{f}, \rho)$ of linear combinations of maps

$$\eta \mapsto \eta(A_1) \cdots \eta(A_k), \quad k \leq n,$$

$A_1, \dots, A_k \subset \mathbb{R}^d$. Contains maps

$$\eta \mapsto \int f_n \mathrm{d}\eta^{\otimes n}.$$

Orthogonal version

$$\mathcal{P}_n(\eta; f_n) = \int f_n \mathrm{d}\eta^{\otimes n} - \text{orthogonal projection onto } \overline{\mathscr{P}_{n-1}}.$$

Theorem

(P_t) consistent, conservative, process (η_t), reversible measure ρ
$(p_t^{(n)})$ compatible n-particle dynamics

$$\mathbb{E}_{\eta_0}\left[\mathcal{P}_n(\eta_t; f_n)\right] = \mathcal{P}_n\big(\eta_0; p_t^{(n)} f_n\big) \qquad \rho\text{-almost all } \eta_0 \in \mathbf{N}_\mathrm{f}.$$

Proof does not need explicit formulas or recurrence relations for polynomials.

Orthogonal polynomials of particular interest when ρ is law of Lévy point process η

$$\mathbb{E}\Big[\exp\Big(-\int f\,\mathrm{d}\eta\Big)\Big] = \exp\Big(\int \sum_{k\in\mathbb{N}} (\mathrm{e}^{-kf(x)} - 1)\, m(k)\alpha(\mathrm{d}x)\Big).$$

$m(k) = \delta_{k,1}$: Poisson point process.

- Schoutens 2000:
  Orthogonal polynomials and stochastic processes.
- Nualart, Schoutens 2000:
  Chaotic and predictable representations for Lévy processes.
- Lytvynov 2004:
  Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes.
  Builds on Berezansky, Mierzejewski. . . .

Free Kawasaki / Poisson-Charlier

$\rho =$ law of Poisson point process with intensity measure λ

$f_n = \mathbb{1}_{A_1}^{\otimes n_1} \otimes \cdots \otimes \mathbb{1}_{A_k}^{\otimes n_k}$, $n_1 + \cdots + n_k = n$, $A_i \subset \mathbb{R}^d$ disjoint,

$$\int f_n \mathrm{d}\eta^{\otimes n} = \eta(A_1)^{n_1} \cdots \eta(A_n)^{n_k},$$

Orthogonal version

$$\mathcal{P}_n(\eta; f_n) = \prod_{i=1}^{k} \mathscr{C}_{n_i}\big(\eta(A_i); \lambda(A_i)\big),$$

where

$$\mathscr{C}_n(x; \alpha) = x^n + \text{lower order terms} \quad (x \in \mathbb{N}_0)$$

orthogonal w.r.t. Poisson law on $\mathbb{N}_0$ with parameter α.

$$\boxed{\mathbb{E}_{\eta_0}\big[\mathcal{P}_n(\eta_t; f_n)\big] = \mathcal{P}_n\big(\eta_0; p_t^{\otimes n} f_n\big).}$$

Application to symmetric inclusion process

$$Lf(\eta) = \iint \big(f(\eta - \delta_x + \delta_y) - f(\eta)\big)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x)$$

Pascal point process (negative binomial point process) $p \in (0,1)$, $\alpha(\mathrm{d}x)$

- $B_1, \dots, B_m \subset \mathbb{R}^d$ disjoint $\Rightarrow \eta(B_1), \dots, \eta(B_m)$ independent.
- $\eta(B) =$ a negative binomial random variable,

$$\mathbb{P}\big(\eta(B) = n\big) = (1-p)^\beta \frac{\beta(\beta+1)\cdots(\beta+n-1)}{n!} p^n, \quad \beta = \alpha(B).$$

Bruss, Rogers SPA '91 *... distinguished role of the Pascal distribution in finding explicit solutions of optimal selection problems based on relative ranks.*

Proposition

Symmetric inclusion process is consistent.

For every $p \in (0,1)$, law $\rho_{p,\alpha}$ of negative binomial point process is reversible for the symmetric inclusion process.

Negative binomial process and Ewens measure

Expectation of functions of negative binomial process

$$\int f \,\mathrm{d}\rho = \left(1-p\right)^{\alpha(E)} \left(f(0) + \sum_{n=1}^{\infty} \frac{p^n}{n!} \int f(\delta_{x_1} + \cdots + \delta_{x_n}) \lambda_n(\mathrm{d}\boldsymbol{x}) \right)$$

Measures $\lambda_1 = \alpha$,

$$\lambda_2(A) = \int \mathbb{1}_A(x_1, x_2)\alpha(\mathrm{d}x_1)\alpha(\mathrm{d}x_2) + \int \mathbb{1}_A(x, x)\alpha(\mathrm{d}x)$$

more generally, λ_n is a sum over set partitions of $\{1, \ldots, n\}$.

Total mass

$$\lambda_n(E^n) = \sum_{\sigma \in \Sigma_n} \theta^{|\sigma|} \prod_{B \in \sigma} (|B| - 1)!, \qquad \theta = \alpha(E)$$

$|\sigma|$ number of blocks in set partition.

Compare: Ewens probability measure on set partitions.

Compatible labelled generator

$$
\begin{aligned}
L_n f_n(x_1,\dots,x_n) = \sum_{i=1}^{n} \Big(\int \big(f(x_1,\dots,\cancel{x_i}\; y,\dots,x_n) - f_n(x_1,\dots,x_n)\big)\alpha(\mathrm{d}y) \\
+ \sum_{i=1}^{n}\sum_{j=1}^{n} \big(f(x_1,\dots,\cancel{x_i}\; x_j,\dots,x_n) - f_n(x_1,\dots,x_n)\big)\Big)
\end{aligned}
$$

Theorem

$(\eta_t)_{t\geq 0}$ *symmetric inclusion process,* $p \in (0,1)$ *fixed, orthogonalization in* $L^2(\rho_{p,\alpha})$

$$\mathbb{E}_{\eta_0}\left[\mathcal{P}_n\big(\eta_t; f_n\big)\right] = \mathcal{P}_n(\eta_0; p_t^{(n)} f_n)$$

for $\rho_{p,\alpha}$ *almost all* η_0*.*

Orthogonal version of $\int f_n \mathrm{d}\eta^{\otimes n} = \eta(A_1)^{n_1}\cdots\eta(A_k)^{n_k}$ is

$$\mathcal{P}_n\big(\eta; f_n\big) = \prod_{i=1}^{k} \mathcal{M}_{n_i}\big(\eta(A_i); p, \alpha(A_i)\big)$$

product of univariate Meixner polynomials.

Summary

How to generalize product dualities with orthogonal polynomials from lattices?

Natural framework:

Orthogonal polynomials from chaos decompositions and Lévy white noise

Still missing:

applications!

For proving scaling limits, analyzing fluctuation fields. . . ??