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Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus

école normale supérieure rennes

Rita Nader
Ecole normale supérieure, Rennes, France

Based on a joint work with Nils Berglund

Bifurcation delay phenomenon

On the slow time scale εt:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x)$$

- Equilibrium branch: $\{x = x^*(t)\}$ where $f(t, x^*(t)) = 0$ for all t
- Stable: if $\partial_x f(t, x^*(t)) < 0$ for all t
- Example: "The dynamic pitchfork bifurcation"

$$f(t,x) = tx - x^3;$$

Bifurcation delay phenomenon:

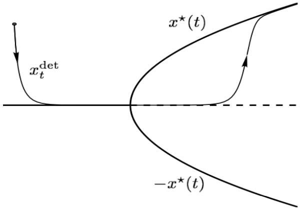

Figure: Solutions of the ODE represented in the (t,x)-plane.

Bifurcation delay in one dimensional SDEs

$$\mathrm{d}x = \frac{1}{\varepsilon} f(t,x)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

- $0 \leqslant \varepsilon, \sigma \ll 1$;
- W_t is a standard Wiener process

In [Berglund & Gentz, Probab Theory Relat Fields, 2002], sample paths

- remain close to the deterministic solution at a distance of order $\sigma\varepsilon^{-1/4}$ up to time $\mathcal{O}(\sqrt{\varepsilon})$ with high probability,
- are unlikely to remain close to 0 after a time of $\mathcal{O}(\sqrt{\varepsilon \log(\sigma^{-1})})$.

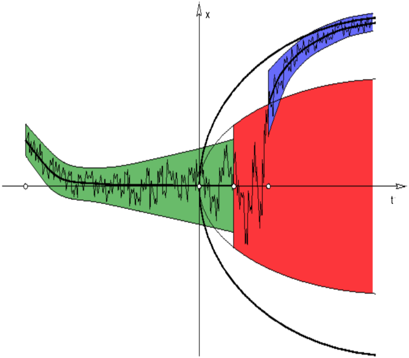

Figure: Sample paths near a pitchfork bifurcation.

Infinite-dimensional stochastic PDEs on $\mathbb{T}^d$

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\left[\Delta\phi(t,x) + F(t,\phi(t,x))\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $\phi(t,x) \in \mathbb{R}$, $t \in I = [0,T] \subset \mathbb{R}_+$, $x \in \mathbb{T}^d = (\mathbb{R}/L\mathbb{Z})^d$, $L > 0$, $d = 1,2$;
- $\varepsilon > 0, \sigma \geqslant 0$;
- $F(t,\phi) = \sum_{j=0}^{n} A_j(t)\phi^j$ for some odd $n \geqslant 3$ and $A_n(t) < 0$ for all $t \in I$;
- $\mathrm{d}W(t,x) = \xi(t,x)\,\mathrm{d}t$ on $I \times \mathbb{T}^d$,
  - ξ space-time white noise: centered, Gaussian, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$;
  - ξ distribution, $\langle\xi,\varphi\rangle \sim \mathcal{N}(0,\|\varphi\|_{L^2}^2)$, $\mathbb{E}[\langle\xi,\varphi_1\rangle\langle\xi,\varphi_2\rangle] = \langle\varphi_1,\varphi_2\rangle$.

Deterministic dynamics

Stable equilibrium branch: $\exists\ \phi^* : I \to \mathbb{R}$ such that $F(t, \phi^*(t)) = 0$ for all $t \in I$ and $a^*(t) = \partial_\phi F(t, \phi^*(t)) < 0$.

Deterministic dynamics near a stable equilibrium branch: There exists a particular solution $\bar{\phi}(t, x)$ such that

$$\|\bar{\phi}(t, \cdot) - \phi^*(t) e_0\|_{H^1} \leqslant C\varepsilon \qquad \forall\, t \in I,\ e_0(x) = \frac{1}{\sqrt{L}}$$

Infinite-dimensional stochastic PDEs on $\mathbb{T}^d$

Write $\psi(t,x) = \phi(t,x) - \overline{\phi}(t,x)$ and Taylor expand:

$$\mathrm{d}\psi(t,x) = \frac{1}{\varepsilon}\big[\Delta\psi(t,x) + a(t)\psi(t,x) + \underbrace{b(t,\psi(t,x))}_{=\mathcal{O}(\psi^2)}\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x),$$

and $a(t) = a^*(t) + \mathcal{O}(\varepsilon)$.

Duhamel formula (variations of constants), if $\psi(0,\cdot) = 0$:

$$\psi(t,\cdot) = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{\bar\alpha(t,t_1)/\varepsilon}\,\mathrm{e}^{[(t-t_1)/\varepsilon]\Delta}\,\mathrm{d}W(t_1,\cdot)}_{\psi_0(t,\cdot)\text{: solution of linearised equation}}$$
$$+ \underbrace{\frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\bar\alpha(t,t_1)/\varepsilon}\,\mathrm{e}^{[(t-t_1)/\varepsilon]\Delta}\,b(t_1,\psi(t_1,\cdot))\,\mathrm{d}t_1}_{\text{treat as a perturbation}}$$

where $\bar\alpha(t,t_1) = \int_{t_1}^t a(u)\,\mathrm{d}u$.

Stochastic resonance in SPDEs on $\mathbb{T}$

In [Berglund & Nader, SPDEs: Analysis and Computations, 2022],

- ▷ Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s-norm for any $s < \frac{1}{2}$
- ▷ At bifurcations, $\phi_\perp = \phi - \int_{\mathbb{T}} \phi \, dx$ remains small in H^s-norm
- ▷ $\sigma < \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
- ▷ $\sigma > \sigma_c$: transition probability per period $\geqslant 1 - e^{-c\sigma^{\frac{3}{4}}/(\varepsilon|\sigma|)}$

Stochastic PDEs on $\mathbb{T}^2$: Wick calculus

- Spectral Galerkin approx: Fourier modes with $|k| \leqslant N$, $k \in \mathbb{Z}^2$
- Variance of ψ_0: $C_N := \sum_{|k|\leqslant N} \frac{1}{\mu_k + 1} \sim \sigma^2 \log N$, $\mu_k = (2\pi)^2 \|k\|^2$
- Wick calculus: $:\phi^m: = H_m(\phi; C_N)$ where H_m Hermite polynomials.
  Example:

$$:\phi^1: = \phi\,, \qquad :\phi^3: = \phi^3 - 3C_N\phi,$$
$$:\phi^2: = \phi^2 - C_N\,, \qquad :\phi^4: = \phi^4 - 6C_N\phi^2 + 3C_N^2\,.$$

- Besov spaces: $\alpha \in \mathbb{R}$ and $p, r \geqslant 1$, $\|\phi\|_{\mathcal{B}^{\alpha}_{p,r}} = \left\| \left\{ 2^{rq\alpha} \|\delta_q \phi\|_{L^p} \right\}_{q\geqslant 0} \right\|_{\ell^r}$
  where $\delta_q \phi(x) := \sum_{k \in \mathcal{A}_q} \phi_k e_k(x)$ and $\mathcal{A}_q = \{k \in \mathbb{Z}^2 : 2^{q-1} \leqslant |k| < 2^q\}$.

Renormalised SPDE

$$\mathrm{d}\phi_N(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi_N(t,x) + {:}F(t,\phi_N(t,x)){:}_{C_N}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_N(t,x)$$

where

- $:F(t,\phi):_{C_N} := \sum_{j=0}^{n} A_j(t){:}\phi^j{:}_{C_N}$
- Its solutions admit a well-defined limit as $N \to \infty$, in appropriate Besov spaces, as proved in [DaPrato & Debussche, Journal Ann. Probab., 2003]

Deviation from the deterministic solution

The difference $\psi = \phi - \bar{\phi}$ satisfies the renormalised SPDE

$$\mathrm{d}\psi(t,\cdot) = \frac{1}{\varepsilon}\Big[\Delta\psi(t,\cdot) + a(t)\psi(t,\cdot) + {:}b(t,\cdot,\psi(t,\cdot)){:}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,\cdot)\;,$$

where ${:}b(t,x,\psi(t,x)){:} = \sum_{j=1}^{n} \hat{A}_j(t,x){:}\psi(t,x)^j{:}\,.$

Theorem 1. "Wick powers of the stochastic convolution"

Let $\psi_0(t,x)$ solution of the linear equation

$$\mathrm{d}\psi_0(t,x) = \frac{1}{\varepsilon}\Big[\Delta\psi_0(t,x) + a(t)\psi_0(t,x)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x),$$

For any $\alpha < 0$ and $m \in \mathbb{N}$, $\exists\ C_m(T,\varepsilon,\alpha)$ and $\kappa_m(\alpha)$, independent of the cut-off N;

$$\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T} \|{:}\psi_0(t,\cdot)^m{:}\|_{\mathcal{B}^{\alpha}_{2,\infty}} > h^m\right\} \leqslant C_m(T,\varepsilon,\alpha)\,\mathrm{e}^{-\kappa_m(\alpha)h^2/\sigma^2}$$

holds for all $h > 0$.

Da-Prato Debussche Trick and stoch. dynamics

Solving the fixed-point argument for $\phi_1 = \psi - \psi_0$ where ϕ_1 satisfies

$$\mathrm{d}\phi_1(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi_1(t,x) + a(t)\phi_1(t,x) + {:}b(t,x,\psi_0(t,x)+\phi_1(t,x)){:}\Big]\,\mathrm{d}t\ ,$$

where

$${:}b(t,x,\psi_0(t,x)+\phi_1(t,x)){:} = \sum_{j=1}^{n} \hat{A}_j(t,x) \sum_{\ell=0}^{j} \binom{j}{\ell} \phi_1(t,x)^{j-\ell} {:}\psi_0(t,x)^{\ell}{:}\,.$$

Theorem 2. "Concentration estimate for ϕ_1"

For any $\gamma < 2$ and $\nu < 1 - \frac{\gamma}{2}$, $\exists\ C(T,\varepsilon), M, \kappa, h_0, \varepsilon_0 > 0$; whenever $\varepsilon < \varepsilon_0$ and $h < h_0\varepsilon^{\nu}$, one has

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} \|\phi_1(t)\|_{\mathcal{B}^{\gamma}_{2,\infty}} > M\varepsilon^{-\nu} h(h+\varepsilon)\Big\} \leqslant C(T,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2}\,.$$

Bifurcation delay in SPDEs

An example of two-dimensional SPDE

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + a(t)\phi(t,x) - {:}\phi(t,x)^3{:}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)\;,$$

where $a(t) = \partial_\phi F(t,\phi^*(t))$ changes sign at a time t^*.

Deterministic dynamics near a pitchfork bifurcation: Solutions attracted by $\phi^*(t) = 0$ for $t < t^*$ remain close to 0 for a time of order 1 beyond the bifurcation time t^*, even though the equilibrium branch has become unstable.

Bifurcation delay in SPDEs

- Change of variables $\phi_1(t,\cdot) = \phi(t,\cdot) - \psi_\perp(t,\cdot)$, where $\psi_\perp$ solves
$$\mathrm{d}\psi_\perp(t,\cdot) = \frac{1}{\varepsilon}\bigl[\Delta_\perp\psi_\perp(t,\cdot) + a(t)\psi_\perp(t,\cdot)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_\perp(t,\cdot).$$
and ϕ_1 satisfies
$$\mathrm{d}\phi_1(t,\cdot) = \frac{1}{\varepsilon}\bigl[\Delta\phi_1(t,\cdot) + a(t)\phi_1(t,\cdot) + {:}F(\psi_\perp,\phi_1){:}\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t,\cdot)\ ,$$
where ${:}F(\psi_\perp,\phi_1){:} = -{:}\psi_\perp^3{:} - 3\phi_1{:}\psi_\perp^2{:} - 3\phi_1^2\psi_\perp - \phi_1^3$.

- Split $\phi_1(t,x) = \phi_1^0(t)e_0(x) + \phi_1^\perp(t,x)$.
It yields a coupled SDE–SPDE system:

$$\mathrm{d}\phi_1^0(t) = \frac{1}{\varepsilon}\bigl[a(t)\phi_1^0(t) - \phi_1^0(t)^3 + F_0(\psi_\perp,\phi_1^0,\phi_1^\perp)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t)\ ,$$
$$\mathrm{d}\phi_1^\perp(t,\cdot) = \frac{1}{\varepsilon}\bigl[\Delta_\perp\phi_1^\perp(t,\cdot) + a(t)\phi_1^\perp(t,\cdot) + {:}F_\perp(\psi_\perp,\phi_1^0,\phi_1^\perp){:}\bigr]\,\mathrm{d}t\ .$$

Stochastic dynamics for $\phi_1^\perp$

- Assume there exists a constant $a_0 > 0$ such that $a(t) \leqslant (2\pi)^2 - a_0$ for all $t \in [0, T]$;
- Besov embeddings $\mathcal{B}^\gamma_{2,\infty} \hookrightarrow \mathcal{B}^{\gamma-1}_{\infty,\infty} = \mathcal{C}^{\gamma-1}$;
- Given $H_0 > 0$, $\tau_0(H_0) = \inf\{t \in [0, T] : |\phi_1^0(t)| > H_0\}$.

Theorem 3. "Concentration estimate for $\phi_1^\perp$"

For any $\gamma < 2$ and $\nu < 1 - \frac{\gamma}{2}$, $\exists\ C(T, \varepsilon), M, \kappa, h_0 > 0$; whenever $h + H_0 \leqslant h_0 \varepsilon^{\nu/2}$, one has

$$\mathbb{P}\left\{ \sup_{t \in [0, T \wedge \tau_0(H_0)]} \|\phi_1^\perp(t)\|_{\mathcal{C}^{\gamma-1}} > M\varepsilon^{-\nu}(h + H_0)^3 \right\} \leqslant C(T, \varepsilon)\, \mathrm{e}^{-\kappa h^2/\sigma^2} .$$

Dynamics of $\phi_1^0(t)$ near a pitchfork bifurcation

Linear equation: $\mathrm{d}\phi^\circ(t) = \frac{1}{\varepsilon} a(t)\phi^\circ(t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t)$

Variance of $\phi^\circ(t)$

$$v^\circ(t) = v^\circ(0) + \frac{\sigma^2}{\varepsilon}\int_0^t \mathrm{e}^{2\alpha(t,t_1)/\varepsilon}\,\mathrm{d}t_1 \asymp \begin{cases} \dfrac{\sigma^2}{|t-t^*|} & \text{for } 0 \leqslant t \leqslant t^* - \sqrt{\varepsilon}\,, \\ \dfrac{\sigma^2}{\sqrt{\varepsilon}} & \text{for } |t-t^*| \leqslant \sqrt{\varepsilon}\,, \\ \dfrac{\sigma^2}{\sqrt{\varepsilon}}\,\mathrm{e}^{2\alpha(t,t^*)/\varepsilon} & \text{for } t \geqslant t^* + \sqrt{\varepsilon}\,. \end{cases}$$

Define sets

$$\mathcal{B}_-(h_-) = \left\{(t,\phi_1^0) \in [0, t^* + \sqrt{\varepsilon}] \times \mathbb{R} \colon |\phi_1^0| \leqslant \frac{h_-}{\sigma} v^\circ(t)\right\},$$

$$\mathcal{B}_+(h_+) = \left\{(t,\phi_1^0) \in [t^* + \sqrt{\varepsilon}, T] \times \mathbb{R} \colon |\phi_1^0| \leqslant \frac{h_+}{\sqrt{a(t)}}\right\}.$$

Dynamics of $\phi_1^0(t)$ near a pitchfork bifurcation

Theorem 4. "Behaviour of $\phi_1^0(t)$"

$\exists\ M, \varepsilon_0, h_0$; for any $\varepsilon < \varepsilon_0$ and $h_- \leqslant h_0 \varepsilon^{1/2}$, and any $t \leqslant t^* + \varepsilon^{1/2}$, one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}_-(h_-)} \leqslant t\Big\} \leqslant C(t,\varepsilon) \exp\left\{-\frac{h_-^2}{2\sigma^2}\left[1 - \mathcal{O}(\sqrt{\varepsilon}) - \mathcal{O}\left(\frac{h_-^2}{\varepsilon}\right)\right]\right\},$$

where $C(t,\varepsilon) = \mathcal{O}(\alpha(t)/\varepsilon^2)$. Furthermore, for $h_+ = \sigma \log(\sigma^{-1})^{1/2}$ and any $t \geqslant t^* + \varepsilon^{1/2}$, one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}_+(h_+)} \geqslant t\Big\} \leqslant \frac{h_+}{\sigma} \exp\left\{-\kappa \frac{\alpha(t,t^*)}{\varepsilon}\right\} + C(t,\varepsilon)\, \mathrm{e}^{-\kappa \log(\sigma^{-1})/\sqrt{\varepsilon}},$$

for a constant $\kappa > 0$.

Some references

- Berglund, Nils & Gentz, Barbara *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probability Theory and Related Fields, (2002)
- Da Prato, Giuseppe & Debussche, Arnaud *Strong solutions to the stochastic quantization equations*, Journal Ann. Probab. (2003)
- Berglund, Nils & Nader, Rita *Stochastic resonance in stochastic PDEs*, Stochastics and Partial Differential Equations: Analysis and Computations, (2022)
- Berglund, Nils & Nader, Rita *Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus*, Preprint, November 2022, arXiv:2209.15357

Thank you for your attention!