HEAT KERNELS ASSOCIATED WITH ROOT SYSTEMS AND THE DYSON AND DUNKL PROCESSES

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ANALYSIS ON ROOT SYSTEMS: THE SETTING

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root systems

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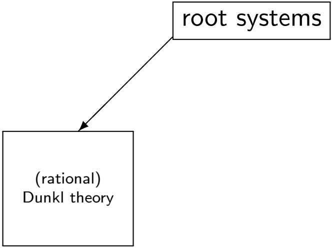

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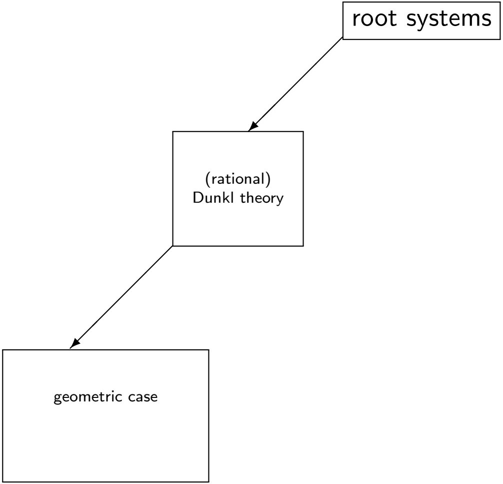

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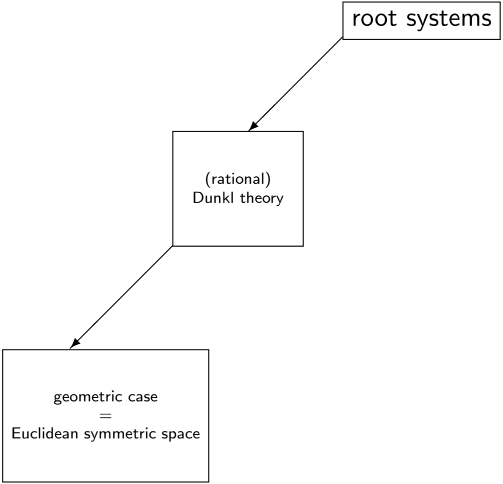

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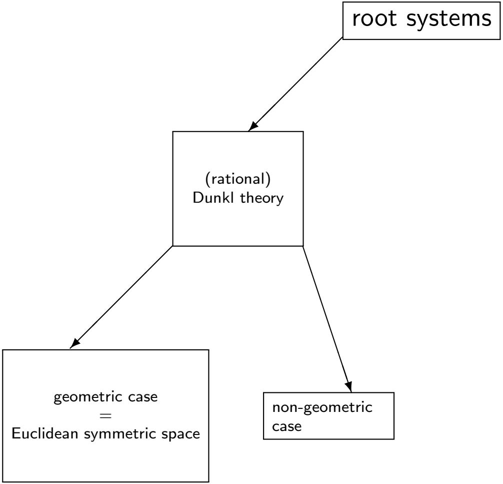

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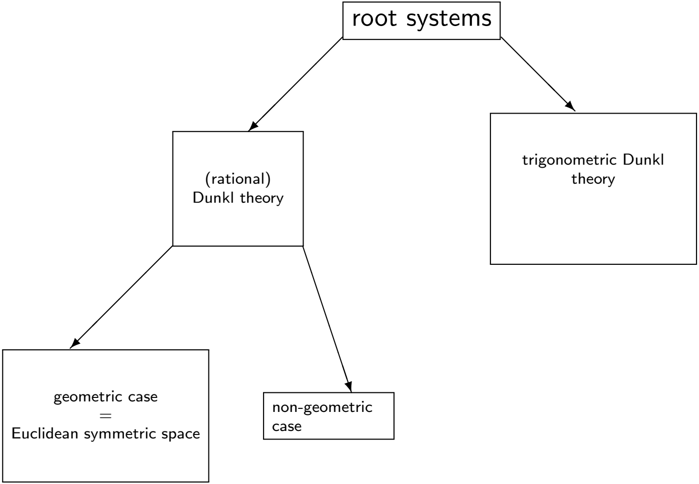

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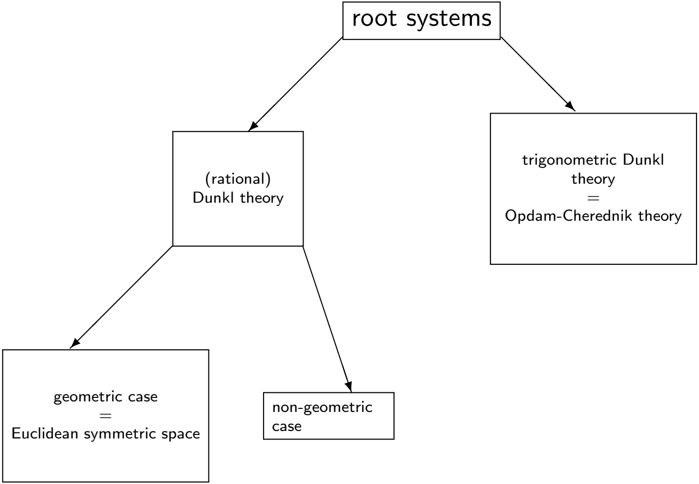

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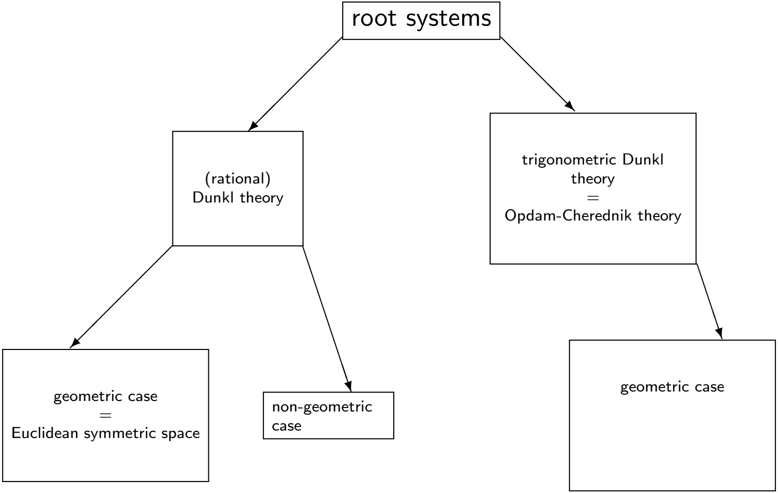

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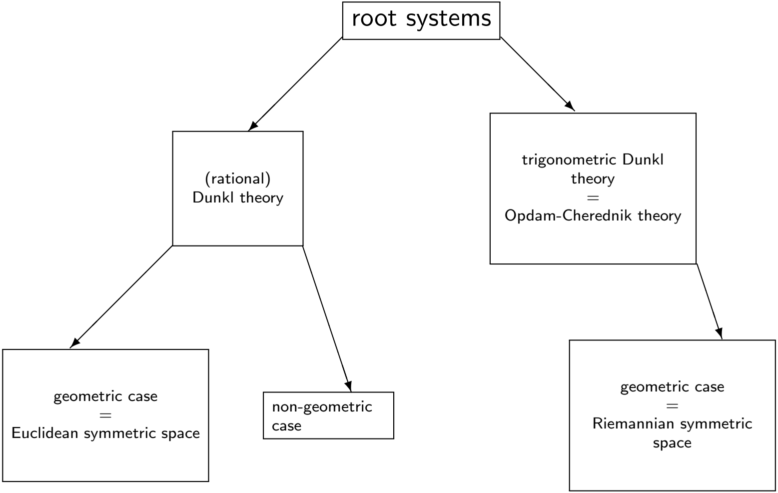

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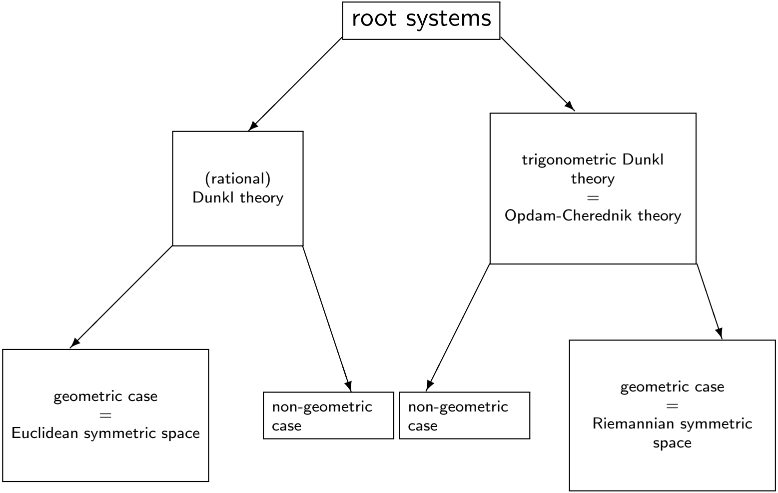

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Roots α are some "very symmetrically chosen" non-zero vectors of $\mathbf{R}^d$.

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The hyperplane

$$H_\alpha = \{x \in \mathbf{R}^d \mid \quad \alpha(x) = \langle \alpha, x \rangle = 0\}$$

orthogonal to α.

REDUCED ROOT SYSTEMS IN R AND R^2

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A_1

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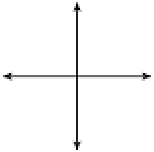
A_1



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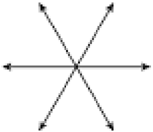

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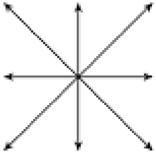

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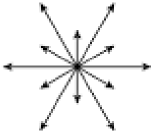

G_2

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For A_{r-1} in $\mathbf{R}^d$, the symmetries $\sigma_{\mathbf{e}_i - \mathbf{e}_j}$, $i \neq j$, $i, j \leq r+1$, are the transpositions of the elements x_i and x_j of the vector x.

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The set of reflecting hyperplanes H_α divides $\mathbf{R}^d$ into connected open components called Weyl chambers (C^+ is one of them).

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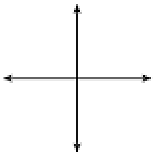

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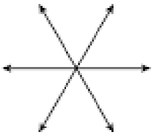

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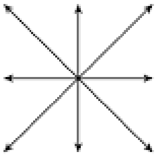

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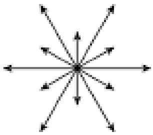

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with $E_k(0,Y)=1$. We study $\psi_\lambda(e^X) = \frac{1}{|W|}\sum_{w\in w} E_k(w\cdot X,\lambda)$.

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- X_t jumps from x to wx, $w \in W$.

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The process $X_t^W := \Pi(X_t)$ is called the radial (W-invariant) Dunkl process or multidimensional Bessel process. X_t^W is a **continuous diffusion** with generator

$$\mathcal{L}_k^W u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha\in\Sigma^+} k(\alpha)\frac{\langle\alpha, \nabla u(x)\rangle}{\langle\alpha, x\rangle}$$

RADIAL (W-INVARIANT) DUNKL PROCESS

Multidimensional Bessel processes

Let Π be the canonical W-projection on the positive Weyl chamber $\overline{C^+}$ (i.e. $\Pi(x)$ is the unique point of the orbit Wx which lies in $\overline{C^+}$).

The process $X_t^W := \Pi(X_t)$ is called the radial (W-invariant) Dunkl process or multidimensional Bessel process. X_t^W is a **continuous diffusion** with generator

$$\mathcal{L}_k^W u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha\in\Sigma^+} k(\alpha)\frac{\langle\alpha, \nabla u(x)\rangle}{\langle\alpha, x\rangle}$$

Theory of Dunkl processes: Gallardo-Yor, Schapira, Demni, Chibiryakov, Voit, Gallardo-Rejeb,...

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For example, if $\Sigma = A_{r-1}$ then $\pi(x) = \prod_{i>j}(x_i - x_j)$, the Vandermonde determinant.

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The generator of X_t^W is then

$$\Delta^W u := \mathcal{L}_{k\equiv 1}^W u = \frac{1}{2}\Delta u + \frac{\langle \nabla \pi, \nabla u \rangle}{\pi} = \pi^{-1}\,\frac{1}{2}\Delta^{\mathbf{R}^r}(\pi\, u).$$

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When a probabilist looks at the last formula, they see in it the generator of the Doob h-transform with the excessive (here harmonic) function $h = \pi(x)$.

For the root system A_{r-1} on $\mathbf{R}^r$, the operator Δ^W is the generator of the Dyson Brownian Motion on $\mathbf{R}^r$, *i.e.* defined as the r Brownian independent particles $B_t^{(1)}, \ldots, B_t^{(r)}$ conditioned not to collide.

DYSON BROWNIAN MOTION IN THE WEYL CHAMBER $\overline{C^+}$ $(k(\alpha) \equiv 1)$

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$$p_t^{\mathrm{D}}(X, Y) = \frac{\pi(Y)}{\pi(X)} p_t^{\mathrm{killed}}(X, Y),$$

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We have

$$p_t^{\mathrm{killed}}(X, Y) = \det(g_t(x_i, y_j))$$

where g_t the classical 1-dimensional heat kernel (Karlin, MacGregor).

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The only difference with the invariant Dunkl case $k = 1$ is that the invariant measure $\pi^2(Y)\,dY$ is used in Dunkl analysis, but it does not appear for the integral kernels in the Dyson Brownian Motion case.

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The Brownian motion in the Weyl chamber $\overline{C^+}$ was studied by Grabiner (IHP 1999), Biane, Bougerol, O'Connell (Duke 2005) and many others.

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The kernels are considered with respect to the Dunkl weight function $\omega_k(Y) = \prod_{\alpha \in \Sigma^+} |\langle \alpha, Y \rangle|^{2\,k(\alpha)}$ on $\mathbf{R}^d$.

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$$N(x,y) = \int_0^\infty p_t(x,y)\, dt.$$

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Conjecture

The above estimates are always true, with the corrections: $\alpha \in \Sigma^{++}$ = set of undivisible positive roots, the power $k(\alpha)+k(2\alpha)$ in the place of $k(\alpha)$.

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THE HEAT KERNEL IN THE DUNKL SETTING

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$$p_t(X, Y) = C_k \, t^{-\frac{d}{2}-\gamma} \, e^{\frac{-|X|^2-|Y|^2}{2t}} \, E_k\left(X, \frac{Y}{t}\right).$$

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In the Weyl-invariant setting:

$$p_t^W(X, Y) = C_k\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{2t}}\, \psi_X\left(\frac{Y}{t}\right).$$

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which is equivalent to: for X, $Y \in \overline{\mathfrak{a}^+}$

$$p_t^W(X,Y) \asymp t^{-\frac{d}{2}}\, e^{\frac{-|X-Y|^2}{2t}} \prod_{\alpha\in\Sigma^{++}} \frac{1}{(t+\alpha(X)\,\alpha(Y))^{k(\alpha)+k(2\alpha)}}.$$

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Compare with the estimates of J.P. Anker, J. Dziubański, A. Hejna (2019-2022) for $p_t(X,Y)$, in the general case.

QUESTION: Could the Conjecture be true in general for $p_t^W(X,Y)$?

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$$\psi_\lambda(e^X) \asymp e^{\lambda(X)} \prod_{i<j} \frac{1}{(1+(\lambda_i-\lambda_j)(x_i-x_j))^k}$$

where $X = \mathrm{diag}[x_1, \ldots x_{n+1}]$, $\lambda(X) = \sum_{j=1}^{n+1} \lambda_j\, x_j$, $x_j \geq x_{j+1}$ and $\lambda_j \geq \lambda_{j+1}$ whenever $i < j$.

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For the heat kernel, this becomes

$$p_t^W(X,Y) \asymp t^{-\frac{d}{2}}\, e^{\frac{-|X-Y|^2}{2t}} \prod_{i<j} \frac{1}{(t+(x_i-x_j)(y_i-y_j))^k}$$

when $x_j \geq x_{j+1}$ and $y_j \geq y_{j+1}$ for $i < j$.

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$\psi_\lambda(e^X) = e^{\lambda(X)}$ if $n = 1$ and

$$\psi_\lambda(e^X) = \frac{\Gamma(k\,(n+1))}{(\Gamma(k))^{n+1}}\, e^{\lambda_{n+1}\,\sum_{r=1}^{n+1} x_r}\, \pi(X)^{1-2\,k} \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \left[\prod_{i=1}^{n}\left(\prod_{j=1}^{i}(x_j - y_i)\prod_{j=i+1}^{n+1}(y_i - x_j)\right)\right]^{k-1} \prod_{i<j\le n}(y_i - y_j)\, dy_1 \cdots dy_n$$

where $\lambda_0(U) = \sum_{r=1}^{n}(\lambda_r - \lambda_{n+1})\, u_k$ and $\pi(X) = \prod_{i<j\le n+1}(x_i - x_j)$.

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The rank 1 case is very representative of the general case.

Next step:

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$$I^{(n)} = \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^{n} (\lambda_i-\lambda_{n+1})\,(x_i-y_i)} \left(\prod_{i\le j\le n} (x_i - y_j) \prod_{i<j\le n+1} (y_i - x_j) \right)^{k-1} \prod_{i<j\le n} \frac{y_i - y_j}{(1+(\lambda_i-\lambda_j)(y_i-y_j))^k}\, dy_1 \ldots dy_n$$

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$$\left(\prod_{i\leq j\leq n}(x_i-y_j)\prod_{i<j\leq n+1}(y_i-x_j)\right)^{k-1}$$

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. . . to be proven!

W-INVARIANT DUNKL NEWTON KERNEL

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It is the fundamental solution of the Cauchy problem

$$\Delta^W u = f$$

where f is given and $|u(x)| \to 0$ as $x \to \infty$.

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Consider a root system of type A. For $X,\, Y \in \overline{\mathfrak{a}^+}$ and for fixed $d \geq 3$ and $k > 0$, we have

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We prove a similar result in the case $d=2$.

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We use the formula

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Lemma

Suppose $k>0$, $a\geq 0$, $b_i\geq 0$, $a+b_i>0$, $i=1,\ldots,m$ and $N>k\,m-1$. Then

$$J := \int_0^\infty \frac{u^N\,e^{-u}\,du}{\prod_{i=1}^m (a+b_i\,u)^k} \asymp \frac{1}{\prod_{i=1}^m (a+b_i)^k}.$$

HEAT SEMIGROUPS FOR FRACTIONAL POWERS OF Δ_k^W

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Let $s \in (0, 2)$. The fractional powers of the W-invariant Dunkl Laplacian

$$(-\Delta_k^W)^{s/2}$$

are the infinitesimal generators of semigroups $(h_t^W(X, Y))_{t \geq 0}$, called *W-invariant Dunkl s-stable semigroups*.

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- in the Dunkl context by Jedidi (2021), Rejeb (2021), Luks (2022).

ESTIMATES OF s-STABLE DUNKL HEAT SEMIGROUPS

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Theorem (PG, PS 2022)

Consider the W-invariant Dunkl Laplacian in the A_n case with multiplicity $k > 0$. Then for $X, Y \in \overline{\mathfrak{a}^+}$,

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$$h_t^{\mathbf{R}^d}(X,Y) \asymp \min\left\{\frac{1}{t^{d/s}}, \frac{t}{|X-Y|^{d+s}}\right\} \asymp \frac{t}{(t^{2/s} + |X-Y|^2)^{(d+s)/2}}.$$

is the s-stable rotationally invariant semigroup on $\mathbf{R}^d$, with generator $(-\Delta)^{s/2}$ (Blumenthal-Getoor)

PROOF

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We have

$$h_t^W(X, Y) = \int_0^\infty p_u^W(X, Y)\, \eta_t(u)\, du$$

where $\eta_t(u)$ is the density of the $s/2$-stable subordinator, i.e. of a positive Lévy process $(Y_t)_{t>0}$ with the Laplace transform

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Upper estimates may be deduced from our estimates of the W-invariant Dunkl heat kernel and from recent results on estimates for subordinanted processes [Grzywny, Trojan (2021)]

OPDAM-CHEREDNIK DERIVATIVES AND LAPLACIAN

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For fixed $Y \in \mathbf{R}^d$, the Opdam-Cherednik kernel $G_k(\cdot, \cdot)$ is then the only real-analytic solution to the system

$$D_\xi(k)|_X \, G_k(X, Y) = \langle \xi, Y \rangle \, G_k(X, Y), \ \forall \xi, X \in \mathbf{R}^d$$

with $G_k(0, Y) = 1$.

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with $G_k(0, Y) = 1$. We study $\phi_\lambda(e^X) = \frac{1}{|W|} \sum_{w \in w} G_k(w \cdot X, \lambda)$.

Opdam-Cherednik stochastic process has $\mathcal{L}$ as generator.

CURVED RIEMANNIAN SYMMETRIC SPACES ARE "INCLUDED" IN W-INVARIANT OPDAM-CHEREDNIK ANALYSIS: $k = 1/2, 1, 2$

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CHALLENGE: HEAT KERNEL ESTIMATES IN THE CURVED SYMMETRIC / DUNKL TRIGONOMETRIC CASE

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$$P_t^W(X, Y) \asymp t^{-\frac{d}{2}} e^{\frac{-|X-Y|^2}{2t}} e^{-|\rho|^2 t/2} e^{-\rho(X+Y)} \prod_{\alpha>0} \frac{(1+\alpha(X))(1+\alpha(Y))}{t+\alpha(X)\alpha(Y)}.$$

PROOF

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We use the relationship between the heat kernel $p_t^W(X, Y)$ in the flat case and the heat kernel $P_t^W(X, Y)$ in the curved case (valid only in the complex case $k = 1$):

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where $\quad \pi(Y) = \prod_{\alpha>0}\alpha(Y), \quad \delta(X) = \prod_{\alpha>0}\sinh^2\alpha(X).$

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$$P_t^W(X, Y) = e^{-|\rho|^2 t/2} \frac{\pi(X)\,\pi(Y)}{\delta^{1/2}(X)\,\delta^{1/2}(Y)}\, p_t^W(X, Y).$$

where $\quad \pi(Y) = \prod_{\alpha>0} \alpha(Y), \quad \delta(X) = \prod_{\alpha>0} \sinh^2 \alpha(X).$

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SPECIAL CASE: A RANK 1 RESULT

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Corollary

Consider the complex hyperbolic space, isomorphic to the 3-dimensional real hyperbolic space $\mathrm{H}^3(\mathbf{R})$.

$$P_t(X,Y) \asymp t^{-\frac{1}{2}}\, e^{\frac{-|X-Y|^2}{2t}}\, e^{-t/2}\, e^{-(X+Y)/2}\, \frac{(1+X)\,(1+Y)}{t+XY},$$

$$X, Y, t \geq 0$$

CONJECTURE FOR SHARP ESTIMATES OF $p_t^w(x, y)$ IN THE W-INVARIANT OPDAM-CHEREDNIK TRIGONOMETRIC CASE

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$$p_t^W(X,Y) \asymp t^{-\frac{d}{2}}\, e^{\frac{-|X-Y|^2}{2t}}\, e^{-|\rho|^2 t/2}\, e^{-\rho(X+Y)} \prod_{\alpha\in\Sigma^{++}} (1+\alpha(X))\,(1+\alpha(Y)) \frac{(t+1+\alpha(X+Y))^{k(\alpha)+k(2\alpha)-1}}{(t+\alpha(X)\,\alpha(Y))^{k(\alpha)+k(2\alpha)}}.$$

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This Conjecture is compatible with the result of Anker and Ostellari (2003) for $P_t(X,0)$ on the hyperbolic spaces and by Schapira (2015) in Opdam-Cherednik setting including the curved symmetric spaces M ($n = \dim(M)$):

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PG and PS proved (2023) the conjecture for A_1 using a technique developed by Anker and Ostellari (2003) and also used by Schapira (2018).

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Reference: P. Graczyk and P. Sawyer. *Sharp estimates for the Opdam-Cherednik W-invariant heat kernel for the root system* A_1 (2022), arXiv:2304.07009, 1–17.

CONJECTURE FOR SHARP ESTIMATES OF $p_t^W(x, y)$ IN THE W-INVARIANT OPDAM-CHEREDNIK TRIGONOMETRIC CASE

Basic idea: use the Weak parabolic minimum principle for unbounded domains and a subdivision of the domain in regions:

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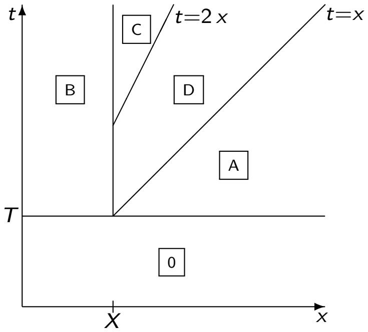

THE END!

THANK YOU FOR YOUR ATTENTION