Asymptotic behavior of a one-dimensional avalanche model through a particular stochastic process

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Aim: to investigate the invariant behaviour of the stochastic differential equations of coagulation-fragmentation

Motivation: Application to avalanches, a real problem

An avalanche model

An avalanche model as **an interacting particle system** with values in $\{0,1\}^{\mathbb{Z}}$ (cf. [X. Bressaud, N. Fournier, *Annals of Probab.*, 2009]) :

- Let $\Gamma = ((\Gamma_t(i))_{t\geq 0})_{i\in\mathbb{Z}}$ be an independent family of Poisson processes with rate 1.
- We assume that on each site $i \in \mathbb{Z}$, the flakes are falling following $(\Gamma_t(i))_{t\geq 0}$ and the birth flakes follows Poisson processes with rate 1.
  - If a flake falls on a vacant site i of $\mathbb{Z}$, i.e. $i = 0$, this site becomes occupied, $i = 1$.
  - If a flake falls on an occupied site $i \in \mathbb{Z}$, **an avalanche starts**: the whole connected component of occupied sites around i becomes vacant.

The **avalanche process** $((\mathbf{y}_t(i))_{t\geq 0})_{i\in\mathbb{Z}}$ is defined for $t \geq 0$ and $i \in \mathbb{Z}$, as:

$\mathbf{y}_t(i) = 1$ if the site i is occupied at the moment t, or
$\mathbf{y}_t(i) = 0$ if the site i is vacant at time t.

The state space is

$$E := \{\mathbf{y} \in \{0,1\}^{\mathbb{Z}} \text{ such that } \liminf_{i\to-\infty} \mathbf{y}(i) = \liminf_{i\to\infty} \mathbf{y}(i) = 0\}.$$

A related binary coagulation-fragmentation model for the avalanche model

- The fragmentation / coagulation phenomenon for an infinite particles system.
- Each particle is characterised by its size and, at some random time, it can coagulate or fragment into two particles.
- $c(i,t) \geq 0$: the density of i clusters at the time t in the system, solution of the **coagulation-fragmentation equation (C-FE):**

$$\begin{cases} \dfrac{\partial}{\partial t}c(i,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{i-1}(K(i-j,j)c(i-j,t)c(j,t) - F(i-j,j)c(i,t)) \\ \qquad\qquad - \displaystyle\sum_{j\in\mathbb{N}^*}(K(i,j)c(j,t)c(i,t) - F(i,j)c(i+j,t)) \\ c(i,0) = c_0(i), \text{for } (i,t) \in \mathbb{N}^* \times \mathbb{R}_+. \end{cases}$$

$K(i,j)$: the rate of coagulation of two particles with masses i and j to form a cluster with mass $i+j$,

$F(i,j)$: the fragmentation rate of a cluster $i+j$ breaks up into two clusters with masses i and j,

Probabilistic model for the avalanche

Properties

• At some random time a particle can splits into two particles or it can coagulate with another one

• *The total mass will be preserved*:
$\sum_{i\geq 1} ic(i,t) = \sum_{i\geq 1} ic_0(i)$ for all $t \geq 0$, and $\sum_{i\geq 1} ic_0(i) < \infty$.

Therefore
$Q_t(\mathrm{d}x) = \sum_{i=1}^{+\infty} ic(i,t)\delta_i(\mathrm{d}x)$ is a probability measure on $\mathbb{N}$ for each t.

Aim:

1. To construct a jump Markov process $(X_t)_{t\geqslant 0}$ such that
$$\mathbb{P}(X_t \in \mathrm{d}x) = Q_t(\mathrm{d}x).$$

This process will describe the evolution of the size of a typical particle in the coagulation-fragmentation system.

2. A numerical approximation for the steady state of the process and a convergence result

Hypothesis

• The **coagulation kernel** $K : G \times G \mapsto \mathbb{R}_+$ is a continuous symmetric map. There exists a constant C such that for all $x, y \in G$

$$K(x, y) \leq C(1 + x + y).$$

There exists a continuous nonnegative function $\phi : G \to [1, \infty)$ such that $x \to \phi(x)/x$ is nonincreasing on G, and for all $x, y \in G$,

$$0 \leq K(y, x) = K(x, y) \leq \phi(x)\phi(y).$$

• The **fragmentation kernel** $F : G \times G \mapsto \mathbb{R}_+$ is a continuous symmetric map. The function $\Psi : G \mapsto \mathbb{R}_+$ defined by

$$\Psi(0) = 0, \ \Psi(x) = \frac{1}{x} \int_0^x y(x - y)F(y, x - y)\mathrm{d}y, \ \text{for } x > 0,$$

is continuous.

A related coagulation-fragmentation equation to the avalanche

- We consider the coagulation and fragmentation kernels: for all $i, j \in \mathbb{N}^*$ we take $K(i,j) = 2$ and

$$F(i,j) = \begin{cases} i+j-1, & \text{if } i = 1 \text{ or } j = 1, \\ 0, & \text{if } i \neq 1 \text{ and } j \neq 1. \end{cases}$$

The particularity of the fragmentation kernel is that a particle i could be only split into two particles one of mass 1 and the other one of mass $i-1$.

The coagulation-fragmentation equation related to the avalanche model is :

$$\begin{cases} \dfrac{\partial}{\partial t} c(i,t) = \dfrac{1}{m_0(t)} \displaystyle\sum_{j=1}^{i-1} c(i-j,t)c(j,t) - (i+1)c(i,t) + ic(i+1,t) \\ c(i,0) = c_0(i), \qquad \text{with } m_0(t) = \sum_{i \geq 1} c(i,t). \end{cases}$$

The interpretation of the avalanche in relation with coagulation-fragmentation model

- We say that two neighbour edges $(i-1, i)$ and $(i, i+1)$ belong to the same particle if $\mathbf{y}_t(i) = 1$.
- We assume that each edge has a mass equal to 1.
- A particle of mass i contains i edges and $i-1$ occupied sites.
- (i, j) belongs to a particle with mass 1 if and only if $\mathbf{y}(i) = \mathbf{y}(j) = 0$.
- For a configuration $\mathbf{y} \in E$ and for $i \in \mathbb{N}$, we assume that there exists the average number of particles with mass i per unit of length

$$c(i, \mathbf{y_t}) := \lim_{n \to \infty} \frac{\text{number of particles with mass } i \text{ in } [-n, n]}{2n+1}.$$

- Based on the invariance by translation of the model, we admit

$$c(i,t) := c(i,\mathbf{y}_t) = \frac{1}{i}\mathbb{P}[\text{ the edge } (0,1) \text{ belongs to a particle with mass } i \text{ in } \mathbf{y}_t].$$

- The family $(c(\cdot,t))_{t\geq 0} = (c(i,t))_{i\geq 0, t\geq 0}$ would also satisfy $\sum_{i\geq 0} ic(i,t) = 1$ for all $t \geq 0$, and

$$\begin{cases} \dfrac{\partial}{\partial t}c(1,t) = -2c(1,t) + \displaystyle\sum_{i\geq 1} ic(i+1,t), \\ \dfrac{\partial}{\partial t}c(i,t) = -2c(i,t) - (i-1)c(i,t) + \\ \qquad \frac{1}{m_0(t)}\sum_{j=1}^{i-1} c(i-j,t)c(j,t) + ic(i+1,t) \quad \text{for all } i \geq 2. \end{cases} \tag{0.1}$$

The interpretation of the specific coagulation-fragmentation eq. in relation with the avalanche

The first equation counts the particles with mass 1 (the isolated edge).

$$\frac{\partial}{\partial t}c(1,t) = -2c(1,t) + \sum_{i\geq 1} ic(i+1,t)$$

- a particle with mass 1 **disappears** when a snowflake falls on an isolated edge and it coagulates at a constant rate 1 with its two neighbours, $K \equiv 2$.

- a particle with mass 1 **appears** if a flake falls on a particle of mass $i+1$ and an avalanche starts: it breaks into two particles of masses 1 and i, $F(i,1)$.

In the second equation, for all $i \geq 2$,

$$\frac{\partial}{\partial t}c(i,t) = -2c(i,t) - (i-1)c(i,t) + \frac{1}{m_0(t)}\sum_{j=1}^{i-1} c(i-j,t)c(j,t) + ic(i+1,t)$$

- the particles of mass $i \geq 2$ which **disappear** after becoming larger with rate 2, when a flake falls on one of its extremities.

- the particles of mass i which **disappear** when a flake falls on a particle with mass i and an avalanche occurs with the rate $F(i-j,j)$

- the particles of mass i which **appear** when a flake falls on one extremity of a particle of mass j, which is the extremity of a particle of mass $i-j$ and they coagulate with the constant rate 2.

- the particles of mass i which **result** after the splitting of a larger particle $i+1$ into two smaller particles, according to $F(i,1)$.

Study the **equilibrium properties of the avalanche process** **y** by using the steady state of the system (0.1) and an adapted stochastic model

Proposition

The system of equations (0.1) *admits a unique steady state denoted by* $c = (c(i))_{i\geq 1}$, *that is:*
(i) $c(i) \geq 0$ *for each* $i \geq 1$
(ii) $\sum_{i\geq 1} ic(i) = 1$
(iii) $c = (c(i))_{i\geq 1}$ *satisfies*

$$\begin{aligned} 2c(1) &= \sum_{i\geq 1} ic(i+1), \\ (i+1)c(i) &= ic(i+1) + \frac{1}{M_c}\sum_{j=1}^{i-1} c(j)c(i-j) \quad \text{for all} \quad i \geq 2, \end{aligned} \tag{0.2}$$

where $M_c := \sum_{i\geq 1} c(i)$ *is supposed uniquely defined and finite.*

The steady state is given implicitly by (0.2). *We can approximate numerically* $M_c \simeq 0,459134$ *and then the steady state of the system.*

Definition

A stochastic process $(X_t)_{t\geq 0}$ is a **solution of the stochastic differential equation of coagulation-fragmentation** (SDECF) if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that

- X_0 is a $xc_0(\mathrm{d}x)$ distributed random variable,
- $(X_t)_{t\geq 0}$ is a càdlàg G-valued $(\mathcal{F}_t)_{t\geq 0}$ adapted process,
- There exist two independent Poisson measures $N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$ and $M(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ on $[0, +\infty) \times G \times [0, +\infty)$ with intensity measures $\mathrm{d}sQ_s(\mathrm{d}y)\mathrm{d}z$, and $\mathrm{d}s\mathrm{d}y\mathrm{d}z$ respectively, where for $s \geq 0$, Q_s is the distribution of X_s, such that a.s. for all $t \geq 0$

$$\begin{aligned} X_t &= X_0 + \int_0^t \int_0^{+\infty} \int_0^{+\infty} y 1_{\{z \leq \frac{K(X_{s-},y)}{y}\}} 1_{\{y<\infty\}} N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \\ &\quad - \int_0^t \int_0^{+\infty} \int_0^{+\infty} y 1_{\{y\in(0,X_{s-})\}} 1_{\{z \leq \frac{X_{s-}-y}{X_{s-}} F(y, X_{s-}-y)\}} M(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z). \end{aligned}$$

The mass of a typical particle in the system is obtained by adding, at some random Poissonian times, the mass of another typical particle, driven by the K; either by splitting, driven by the F, this mass into two smaller masses.

Numerical approximation for the stochastic differential equation of the coagulation-fragmentation equation

- Under the Hypothesis on F and K, there exists a unique weak solution to the (SDECF), cf. [Fournier, Giet, *J. Stat. Phys.* 2003], in the discrete case [Jourdain, *Markov Processes and Related Fields*, 2003].

Aim

- to approximate **the steady state of the avalanche** by using a numerical approach for the solution of the stochastic differential equation of the coagulation-fragmentation equation (SDECF);

- to give a **convergence result**

Simulation of the coagulation-fragmentation process as a solution of (SDECF)

Step 0: Sampling $X_0^1, \dots, X_0^n$ i.i.d.r.v., $\sim Q_0(\mathrm{d}x) = xc_0(\mathrm{d}x)$.

Step p.1:Sampling a r. v. $y^i \sim \mathcal{U}([0, X^i_{T_{p-1}}])$ for all $i \in \{1, ..., n\}$.

Compute $m^i_{p,f} = \dfrac{X^i_{T_{p-1}} - y^i}{X^i_{T_{p-1}}} F(y^i, X^i_{T_{p-1}} - y^i)$ for all $i \in \{1, ..., n\}$.

Sampling a random variable $S^i_{p,f} \sim \mathcal{E}(m^i_{p,f})$ for all $i \in \{1, ..., n\}$.

$$S_{p,f} = \min_i \{S^i_{p,f}\},\ i_f = {}_i\{S^i_{p,f}\}.$$

Step p.2:Compute $m_{p,c} = \sup_{i,j} \dfrac{K(X^i_{T_{p-1}}, X^j_{T_{p-1}})}{X^j_{T_{p-1}}}$ for all $i, j \in \{1, ..., n\}$.

Set $i_c = i, j_c = j$ the couple for which the sup is realised.
Sampling a random variable $S_{p,c} \sim \mathcal{E}(nm_{p-1,c})$.

Step p.3: Set $S_p = \min\{S_{p,f}, S_{p,c}\}$.

Step p.3f:If $S_p = S_{p,f}$ then $X^{i_f}_{T_{p-1}}$ could fragment.

Sampling a random variable $u \sim \mathcal{U}([0, 1])$; If $u \leq m^{i_f}_{p,f}$, then fragmentation occurs and set

$$X^{i_f}_{T_p} = X^{i_f}_{T_{p-1}} - y^{i_f}; X^k_{T_p} = X^k_{T_{p-1}} \text{ for all } k \neq i_f.$$

Else set $X^k_{T_p} = X^k_{T_{p-1}}$, for all $k \in \{1, \dots, n\}$.

Step p.3c:If $S_p = S_{p,c}$ then a coagulation can occur. At time S_p

the pair (i_c, j_c) could coagulate and sampling $u \sim \mathcal{U}([0, m_{p,c}])$.

If $u \leq \dfrac{K(X^{i_c}_{T_{p-1}}, X^{j_c}_{T_{p-1}})}{X^{j_c}_{T_{p-1}}}$, then the coagulation occurs and set

$$X^{i_c}_{T_p} = X^{i_c}_{T_{p-1}} + X^{j_c}_{T_{p-1}}; X^{k}_{T_p} = X^{k}_{T_{p-1}} \text{ for all } k \neq i_c.$$

Else set $X^k_{T_p} = X^k_{T_{p-1}}$ for all $k \in \{1, ..., n\}$.

Step p.4: Set $T_p = T_{p-1} + S_p$ and for all $k \in \{1, ..., n\}$ and $t \in [T_{p-1}, T_p)$, $X^k_t = X^k_{T_{p-1}}$.

Stop: When $T_p > T$, set, for all $t \in [T_{p-1}, T]$, $X^k_t = X^k_{T_{p-1}}$ for all $k \in \{1, ..., n\}$.

Outcome: The approximated particles mass at time T, $X^k_{T_{p-1}}$ for all $k \in \{1, ..., n\}$.

Numerical results.

In Table 1 we give the values of $c_i, i = 1, \dots, 5$, **the concentration of the particles of mass** i in the system $N = 15$, $T = 100$ and Monte Carlo parameter $M = 10^4$.

Table: Numerical approximations of concentration of particles $c_i, i = 1, \dots, 5$

c_i	Algorithm A1	Value given by Proposition 0.1
c_1	0.2707	0.270433
c_2	0.1003	0.081732
c_3	0.0209	0.042954
c_4	0.0050	0.025178
c_5	0.0013	0.015186

$c_6 \simeq 0.009232$, $c_7 \simeq 0.005625$, $c_8 \simeq 0.003430$, $c_9 \simeq 0.002092$, $c_{10} \simeq 0.001275, \dots..$

So, we can compute $\sum_{i=1}^{50} c_i = 0.459036$, which is a good approximation of M_c.

The convergence of the algorithm

Recursive Algorithm for the (SDECF)

Sampling $X_0^1, \ldots, X_0^n$ i.i.d.r.v., according to $Q_0(\mathrm{d}x) = xc_0(\mathrm{d}x)$.
Set the final time T and the initial $t = 0$;

Set x= $X_0 = (X_0^1, \ldots, X_0^n)$;

While $(t < T)$, do
{
Compute the fragmentation or coagulation time $S_p = \min\{S_{p,f}, S_{p,c}\}$ and the i_f, i_c, j_c as in Step p.1 and Step p.2 from the Algorithm A1.
Set X_s =x for all $s \in [t, (t + S_p) \wedge T]$
Set $t = t + S_p$
If $t \leq T$
{
Choose w uniformly in $[0, 1]$
Set xm=mass(t,w,x)
Set x(i_f)=xm(i_f)-y(i_f), for the fragmentation case
or set x(i_c)=xm(i_c)+xm(j_c), for the coagulation case
}
}

The convergence results

Théorème

Let $T < \infty$. We assume that K, F satisfies the conditions from the Hypothesis and suppose that for all $x, x', y \in \mathbb{N}^*$ if $x \leq x'$ then $K(x, y) \leq K(x', y)$. Denote by C_T the total number of times that Recursive Algorithm takes before ending.
Then

$$\mathbb{E}[C_T] < \infty$$

and the Recursive Algorithm ends a.s. We denote by $(X_t)_{t\in[0,T]}$ be the process constructed by the algorithm. Then $(X_t)_{t\in[0,T]}$ satisfies the stochastic differential equation of coagulation-fragmentation (SDECF).

[O. L-S, M. Deaconu, *Asymptotic behavior of a one-dimensional avalanche model through a particular stochastic process*, preprint, 2023]

Sketch of the proof

(i) For any $r \geq 0$, any $v \in (0,1)$, and any $z \in \mathbb{N}^*$, we denote by $P_{r,v,z}(\mathrm{d}x, \mathrm{d}c)$ the law of a couple of random variables $(X_{r,v,z}, C_{r,v,z})$:

- $C_{r,v,z}$ the (possibly infinite) number of times that the execution of the `mass(r,v,z)`
- $X_{r,v,z}$ is the result of the function `mass(r,v,z)`,i.e. gives the possibility to have a fragmentation or a coagulation procedure.

(ii) For each $r \geq 0$, we denote by

- C_r the (possibly infinite) total number of times that `Recursive Algorithm` calls the function `mass` to obtain X_r.
- Then C_r is a nondecreasing $\mathbb{N} \cup \{\infty\}$-valued process. and since $X_{r,v,z}$ is simulated essentially in the same way as X_r, in law, we have

$$(X_{r,v,z}, C_{r,v,z}) \overset{(d)}{=} (X_r(1_{\{v \leq [K(z,X_r)/X_r]/\lambda(z)\}} 1_{\{X_r < \infty\}} 1_{\text{coag.}} -$$

$$1_{\{X_r \in (0,z)\}} 1_{\{u \leq \frac{z-X_r}{z} F(X_r, z-X_r)\}} 1_{\text{frag.}}, C_r), (0.3)$$

For each $r \in [0, T]$ we denote by Q_r the law of the $\mathbb{N}^* \cup \{\infty\}$-valued r. v. X_r. $(X_r)_{r \in [0,T]}$ is now well-defined as a càdlàg, $\mathbb{N}^* \cup \{\infty\}$-valued process, and X_0 has the distribution $xQ_0(\mathrm{d}x)$.

Let D be the set $\mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. For each $r \in [0, T]$ we have that

- $$X_r = X_0 + \int_0^r \int_0^1 \int_D x M^c(\mathrm{d}s, \mathrm{d}v, d(x,c)) - \int_0^r \int_0^1 \int_D x M^f(\mathrm{d}s, \mathrm{d}u, d(x,c)), \tag{0.4}$$

where $M^c(\mathrm{d}s, \mathrm{d}v, d(x,c))$ and $M^f(\mathrm{d}s, \mathrm{d}u, d(x,c))$ are random integer-valued measures (see [JacSh87] on $[0, T] \times (0,1) \times D$, with compensator $\lambda(X_{s-})\mathrm{d}s\mathrm{d}vP_{s,v,X_{s-}}(\mathrm{d}x, \mathrm{d}c)$, respectively $\mathrm{d}s\mathrm{d}uP_{s,u,X_{s-}}(\mathrm{d}x, \mathrm{d}c)$).

- $$C_r = \int_0^r \int_0^1 \int_D (1+c) M^c(\mathrm{d}s, \mathrm{d}v, d(x,c)) + \int_0^r \int_0^1 \int_D (1+c) M^f(\mathrm{d}s, \mathrm{d}v, d(x, \tag{0.5}$$

- $(X_r)_{r \in [0,T]}$ satisfies (SDECF).

Let us now prove that $\mathbb{E}[C_T] < \infty$.

- For $r \in [0, T]$ let n_r be the distribution of the random variable C_r which is $\mathbb{N} \cup \{+\infty\}$ valued. By using formula (21) we can express (0.5):

$$C_r = \int_0^r \int_{\mathbb{N}} (1 + c)\nu(\mathrm{d}s, \mathrm{d}c), \tag{0.6}$$

where ν is a random integer-valued measure of the form $\nu = \nu^c + \nu^f$, and ν^c and ν^f denote also random integer-valued measures with compensators $\lambda(X_{r-})\mathrm{d}rn_r(\mathrm{d}c)$ and $\mathrm{d}rn_r(\mathrm{d}c)$ respectively.

- Let $A \in (0, +\infty)$, we have:

$$C_r \wedge A \leq \int_0^r \mathrm{d}s((1 + c) \wedge A)\nu(ds, dc). \tag{0.7}$$

By taking expectation and denoting $m_T = \sup_{t\in[0,T]} \mathbb{E}(\lambda(X_t) + X_t)$ we get

$$\mathbb{E}(C_r \wedge A) \leq m_T + m_T \int_0^r \mathbb{E}(C_T \wedge A)\mathrm{d}s. \tag{0.8}$$

- By Gronwall Lemma we deduce that $\mathbb{E}(C_T \wedge A) \leq \gamma_T$, where γ_T is a constant not depending on A, and we take the limit as A goes to infinity.

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