Ergodicity of products of random operators

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Finite dimensional problem

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- $(M_n)_{n\in\mathbb{N}}$ sequence of i.i.d random matrices in $\mathcal{M}_d(\mathbb{R}^+)$
- Define

$$M_{0,n} = M_0 \cdots M_{n-1}.$$

Problem : Almost sure asymptotic estimates of $M_{0,n}$?

Motivations : Mathematical curiosity and application to population modeling, in particular of multitype populations in a fluctuating environment.
Ex : To model a population with d types,

$$\mu_0 \in (\mathbb{R}_+)^d,\ \mu_n = \mu_{n-1} M_{n-1} = \mu_0 M_0 \cdots M_{n-1}.$$

Key quantity in the study of multitype GW process in random environment, used in [Pham, 17], [Peigné, Pham, 23], [Le Page, Peigné, Pham, 18], [Grama, Liu, Pin 21], [Grama, Liu, Pin, 22], ...

Products of random matrices : simple cases

Case 1 : d=1, products of random numbers, $M_e = m_e \in \mathbb{R}_+$.

$$M_{0,n} = m_0 \cdots m_{n-1} = \exp\left(\sum_{i=0}^{n-1} \log(m_i)\right),$$

First order behavior of $M_{0,n}$ determined by the sign of $\mathbb{E}\left(\log m_0\right)$.
Case 2 : $M_i = M$ for all i.

$$M_{0,n} = M^n.$$

Theorem (Perron-Frobenius)

Let $M \in \mathcal{M}_d(\mathbb{R}_+^*)$. Then $\lambda_{PF} = \sup\{|\lambda|, \lambda \in Sp(M)\} > 0$ is a simple eigenvalue of M, with right and left eigenvectors $R, L \in (\mathbb{R}_+^*)^d$. Moreover, there exists $\rho \in (0, \lambda)$ such that

$$M^n = \lambda_{PF}^n RL + O(\rho^n),$$

Note that $\lambda_{PF} = \lim \|M^n\|^{\frac{1}{n}} = \exp\left(\lim \frac{1}{n} \log \|M^n\|\right)$

A full answer in finite dimension

Assumption (H)

Almost surely,

- *for each $n \in \mathbb{N}$, M_n has a non zero coefficient on each row and column*
- $T = \inf\{n \in \mathbb{N} | \forall i, j, M_{0,n}(i,j) > 0\} < \infty$.

When $n \geq T$, there exists a unique triplet (Λ_n, L_n, R_n) of P-F eigenelements of $M_{0,n}$

Theorem (Hennion, 1997)

Let (M_n) be a stationary and ergodic sequence in $\mathcal{M}_d(\mathbb{R}_+)$, satisfying (H). Then it holds, almost surely

$$M_{0,n} \sim \Lambda_n R_n L_n$$

$$(\Lambda_n)^{\frac{1}{n}} \to \lambda = \exp\left(\inf_n \frac{1}{n} \mathbb{E}\left[\log\left(\|M_{0,n}\|\right)\right]\right).$$

There exists random vectors R, L such that $R_n \overset{a.s.}{\to} R$ and $L_n \overset{d}{\to} L$.

This implies, for any $x \in \mathbb{R}_+^d$, $x \neq 0$

$$xM_{0,n} \sim \Lambda_n \langle x, R \rangle L_n$$

An infinite dimensional setup

- $(\mathbb{X}, \mathcal{X})$ measurable space
- $\mathcal{K}^+$ set of generalized Markov kernels $M(x, dy)$, such that
  - For each $x \in \mathbb{X}$, $M(x, dy)$ is a positive finite measure on $\mathbb{X}$
  - For each $A \in \mathcal{X}$ $x \mapsto M(x, A)$ is measurable
  - $\sup_{x \in \mathbb{X}} M(x, \mathbb{X}) < \infty$.
- Joint action of $\mathcal{K}^+$ on signed measures and bounded measurable functions defined by

$$\mu M f = \int \int M(x, dy) f(y) \mu(dx).$$

Example : If $\mathbb{X}$ is countable, typically, $\mathbb{X} = \mathbb{N}$, elements of $\mathcal{K}^+$ are represented by infinte matrices, e.g. : infinite Leslie Matrix

$$M = \begin{pmatrix} f_0(\omega) & s_0(\omega) & 0 & 0 & \dots \\ f_1(\omega) & 0 & s_1(\omega) & 0 & \dots \\ f_2(\omega) & 0 & 0 & s_2(\omega) & \ddots \\ f_3(\omega) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

New tools in infinite dimension

Doeblin minoration for a Markov chain : Let Q be a Markov kernel, ν a probability, and c such that

$$\forall x \in \mathbb{X}, M(x, dy) \geq c\nu(dy),$$

then for any probabilities μ_1, μ_2

$$\|\mu_1 M - \mu_2 M\|_{TV} \leq (1-c)\|\mu_1 - \mu_2\|_{TV}$$

Generalization to non conservative operators : see [Bansaye, Cloez, Gabriel, '19] If there exists a probability measure ν, numbers c, d such that

$$\forall x \in \mathbb{X}, M_{0,1}(x, dy) \geq c\nu(dy) M_{0,1}(x, \mathbb{1}),$$

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{X}, M_{1,n}(x, \mathbb{X}) \leq \frac{1}{d} \int_{x \in \mathbb{X}} M_{1,n}(x, \mathbb{X}) \nu(dx) = \frac{1}{d} \nu M_{1,n} \mathbb{1}.$$

Theorem (L.)

Consider an ergodic sequence $(M_n)_n$ in $\mathcal{K}^+$. Under suitable positivity and moments assumptions, among which a ***Doeblin minoration*** *condition, it holds*

i) *There exists $\tilde{\eta} \in (0,1)$, and almost surely, there exists a random function h on $\mathbb{X}$, such that, for any $\eta \in (0, \tilde{\eta})$, for n large enough,*

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leq \eta^n \langle \mu_1 M_{0,n}, \mathbf{1} \rangle.$$

ii) *Setting $\lambda := \inf_N \frac{1}{N} \mathbb{E}\left[\log \|M_{0,n}\|\right] \in [-\infty, \infty)$, it holds, almost surely, for any μ*

$$\frac{1}{n} \log \langle \mu M_{0,n}, \mathbf{1} \rangle \to \lambda.$$

iii) *There exists a random probability π on $\mathbb{X}$ such that for any μ*

$$\left(\frac{\mu M_{0,n}}{\langle \mu M_{0,n}, \mathbf{1} \rangle} \right)_{n \geq 0} \xrightarrow{d} \pi,$$

Thank you for your attention !