Normal approximation of compound Hawkes functionals

Mahmoud Khabou[1,2], Nicolas Privault[3] and Anthony Réveillac[1]

Institut de Mathématiques de Toulouse[1], Institut Élie Cartan de Lorraine[2], Nanyang Technological University[3]

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Outline

1. Definition of the compound Hawkes process
2. Behaviour for large T
3. Malliavin's calculus for Hawkes processes
4. Bounds on the distance between Hawkes functionals and their Gaussian limit

Definition of the compound Hawkes process

Simple counting processes and their intensities

Definition (Simple counting process)

A stochastic process $(H_t)_{t\in\mathbb{R}_+}$ is called a simple counting process if

- $H_t \geq 0$ for any $t \geq 0$,
- H is non-decreasing,
- H has jumps of size 1.

Giving a simple counting process is equivalent to giving an infinite increasing sequence of jumping times

$$0 < \tau_1 < \tau_2 < \cdots$$

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Definition (The intensity process)

The intensity/rate process $(\lambda_t)_{t\in\mathbb{R}_+}$ is the predictable process defined as

$$\lambda_t \mathrm{d}t = \mathbb{E}[H_{t+\mathrm{d}t} - H_t | \mathcal{F}_{t-}].$$

The simple Hawkes process

Definition

Let H be a counting process. We say that H is a simple Hawkes process if its intensity λ follows the dynamics

$$\lambda_t = \mu + \int_0^{t^-} \phi(t-s)\mathrm{d}H_s = \mu + \sum_{\tau_k < t} \phi(t - \tau_k).$$

Where ϕ is a non-negative integrable kernel such that $\|\phi\|_1 < 1$. The constant $\mu > 0$ is called the baseline intensity.

Definition of the compound Hawkes process

Let $(X_k)_{k\in\mathbb{N}}$ be *i.i.d* of distribution ν. Let H be a Hawkes process and assume $X \perp\!\!\!\perp H$. The sum

$$S_T = \sum_{k=1}^{H_T} X_k, \quad T \geq 0$$

is called a compound Hawkes process.

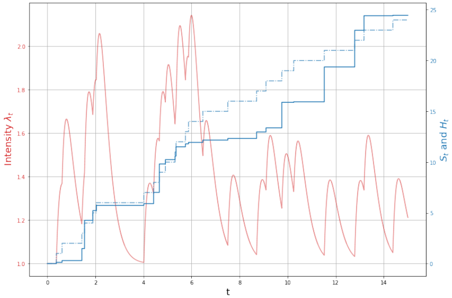

Figure: The compound Hawkes process with a kernel $\phi(s) = 4se^{-4s}$, $\mu = 1$ and $X \sim \mathcal{E}(1)$.

The simple Hawkes process defined as Poisson embedding

The Hawkes process as thinning from a Poisson measure

Let $N(t,\theta)$ be a two-component Poisson measure of intensity $\mathrm{d}t\mathrm{d}\theta$. The following SDE

$$\begin{cases} H_t &= \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\theta\leq\lambda_s} N(\mathrm{d}s,\mathrm{d}\theta), \\ \lambda_t &= \mu + \int_{[0,t)\times\mathbb{R}_+} \phi(t-s)\mathbb{1}_{\theta\leq\lambda_s} N(\mathrm{d}s,\mathrm{d}\theta), \end{cases}$$

has a unique solution such that H is adapted and λ is predictable with respect to the Poisson filtration.

Generalisation: The compound Hawkes process

The Hawkes process as thinning from a Poisson measure

Let $N(t, \theta, x)$ be a three-component Poisson measure of intensity $\mathrm{d}t\mathrm{d}\theta\nu(\mathrm{d}x)$. The following SDE

$$\begin{cases} H_t &= \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} \mathbb{1}_{\theta\leq\lambda_s} N(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x), \\ S_t &= \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} x\mathbb{1}_{\theta\leq\lambda_s} N(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x), \\ \lambda_t &= \mu + \int_{[0,t)\times\mathbb{R}_+\times\mathbb{R}} \phi(t-s)\mathbb{1}_{\theta\leq\lambda_s} N(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x), \end{cases}$$

has a unique solution such that H and S are adapted and λ is predictable with respect to the Poisson filtration.

Illustration for a simple Hawkes process

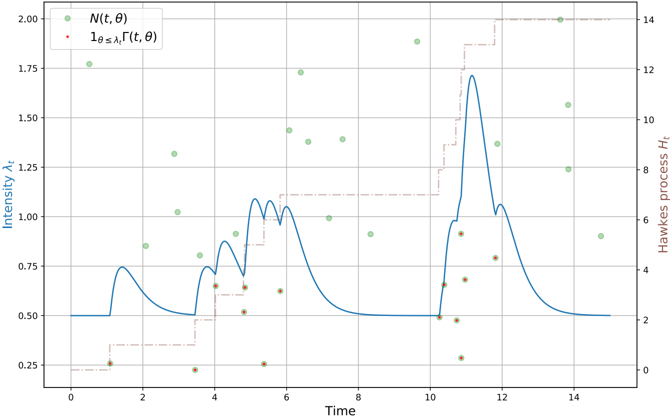

Figure: A simulation with $\phi(s) = 2se^{-3s}$ and $\mu = 0.5$.

Behaviour for large T

The long time behaviour of H

- Do we have a CLT for H?

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- We define the martingale $M_T = H_T - \int_0^T \lambda_t \mathrm{d}t$. For the simple Hawkes process, Bacry *et al.* proved that the normalized martingale

$$\frac{M_T}{\sqrt{T}} \underset{T\to+\infty}{\Longrightarrow} \mathcal{N}(0,\sigma^2),$$

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- What about the compound process?
- Can we quantify the speed of convergence?

Stein's method

- A measure of the distance between two distributions $\mathcal{L}_V$ and $\mathcal{L}_G$ (or V and G) is the Wasserstein metric

$$d_W(V,G) = \sup_{f \in Lip} \left| \mathbb{E}\left[f(V) - f(G)\right] \right|,$$

with $Lip = \{f \in \mathcal{C}^1(\mathbb{R}), \|f'\|_\infty \leq 1\}$.

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- If $G \sim \mathcal{N}(0, \gamma^2)$, C. Stein proved that

$$d_W(V,G) \leq \sup_{f \in \mathcal{F}_W} \left|\mathbb{E}\left[\gamma^2 f'(V) - V f(V)\right]\right|,$$

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- How to obtain a bound if we plug in the normalized martingale F_T?
- For the compound Hawkes process

$$F_T = \frac{S_T - \mathbb{E}[X]\int_0^T \lambda_s \mathrm{d}s}{\sqrt{T}}$$

The bound when F_T is plugged

- We replace V by F_T and we take $G \sim \mathcal{N}(0, \sigma^2)$:

$$\begin{aligned} d_W(F_T, G) &\leq \sup_{f \in \mathcal{F}_W} \left| \mathbb{E}\left[\sigma^2 f'(F_T) - F_T f(F_T)\right]\right| \\ &\leq \sup_{f \in \mathcal{F}_W} \left| \mathbb{E}\left[\sigma^2 f'(F_T) - \delta\left(\left(z_{(t,x)} \mathcal{Z}_{(t,\theta)}\right)_{(t,\theta,x)\in\mathbb{R}_+^2\times\mathbb{R}}\right) f(F_T)\right]\right|, \end{aligned}$$

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- Where δ is the divergence operator with respect to the Poisson measure, defined as

$$\delta(P) = \int_{\mathbb{R}_+^2 \times \mathbb{R}} P_{(t,\theta,x)} \left(N(\mathrm{d}t, \mathrm{d}\theta, \mathrm{d}x) - \mathrm{d}t \mathrm{d}\theta \nu(\mathrm{d}x) \right),$$

and z and $\mathcal{Z}$ are the predictable processes $z_{(t,x)} = x \frac{\mathbb{1}_{t \leq T}}{\sqrt{T}}$ and $\mathcal{Z}_{(t,\theta)} = \mathbb{1}_{\theta \leq \lambda_t}$.

Malliavin's calculus for Hawkes processes

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Definition (Shift operator)

Let $y \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $u \leq t$. The shift operator $\varepsilon_{(u,y)}$ consists of adding an artificial jump/event at time u of height y in the third component:

$$\begin{cases} H_t \circ \varepsilon_{(u,y)} = & H_u + 1 + \int_{(u,t] \times \mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\theta \leq \lambda_s \circ \varepsilon_{(u,y)}} N(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x), \\ S_t \circ \varepsilon_{(u,y)} = & S_u + y + \int_{(u,t] \times \mathbb{R}_+ \times \mathbb{R}} x \mathbb{1}_{\theta \leq \lambda_s \circ \varepsilon_{(u,y)}} N(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x), \\ \lambda_t \circ \varepsilon_{(u,y)} = & \mu + \int_{(0,u)} \phi(t-s) \mathrm{d}H_s + \phi(t-u) \\ & + \int_{(u,t)} \phi(t-s) \mathrm{d}(H_t \circ \varepsilon_{(u,y)}). \end{cases}$$

We extend the operator naturally to any random variable $V \in \sigma(H_s, s \leq t)$.

Definition (Malliavin's derivative)

Let $V \in \sigma(S_v, v \leq t)$. For $u \leq t$, the Malliavin derivative of V is defined as

$$D_{(u,y)} V = V \circ \varepsilon_{(u,y)} - V.$$

Illustration for the intensity on a simple Hawkes process

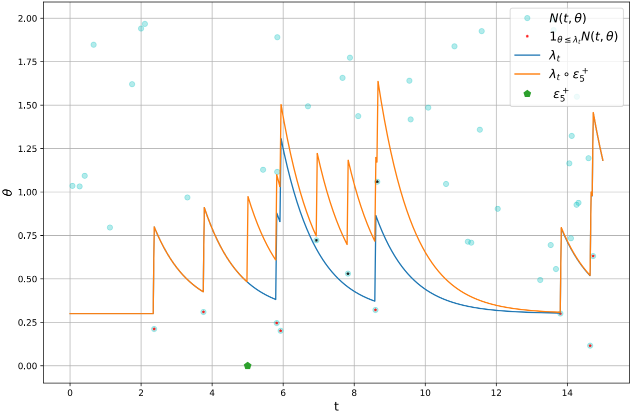

Figure: The effect of adding a jump at time $u = 5$. The kernel is $\phi(s) = 0.5e^{-s}$.

Malliavin derivatives of H and λ

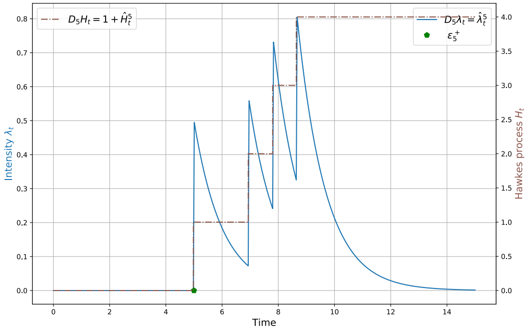

Figure: The processes $\hat{H}_t^u$ and $\hat{\lambda}_t^u$.

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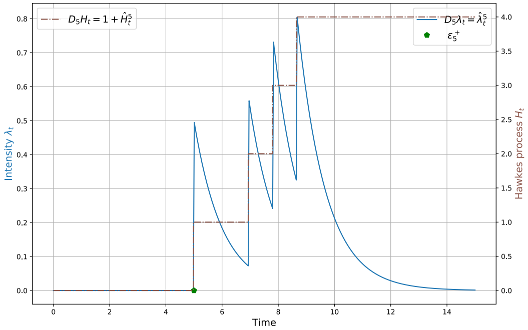

Figure: The processes $\hat{H}_t^u$ and $\hat{\lambda}_t^u$.

Derivative of the normalized martingale

$$D_{(u,y)} F_T = \frac{1}{\sqrt{T}} \left(y + \hat{M}_T^{(u,y)} \right)$$

Integration by parts

Duality

Let $(z_{(t,x)})_{t\geq 0}$ be a predictable process and $V \in \sigma(S_t, t \geq 0)$. It holds that

$$\mathbb{E}[\delta(z\mathcal{Z})V] = \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}} \lambda_t z_{(t,x)} D_{(t,x)} V \mathrm{d}t\nu(\mathrm{d}x)\right].$$

Verification for $X \equiv 1$, $\phi \equiv 0$ and $z_t = \mathbb{1}_{t\leq T}$:

$$\begin{aligned}\mathbb{E}[(H_T - \mu T)f(H_T)] &= \sum_{n=0}^{+\infty} n f(n) e^{-\mu T}\frac{(\mu T)^n}{n!} - \mu T\mathbb{E}[f(H_T)],\\ &= \sum_{n=0}^{+\infty} f(n) e^{-\mu T}\frac{(\mu T)^n}{(n-1)!} - \mu T\mathbb{E}[f(H_T)],\\ &= \mathbb{E}[\mu T\left(f(H_T+1) - f(H_T)\right)].\end{aligned}$$

Bounds on the distance between Hawkes functionals and their Gaussian limit

Reminder

- We would like to bound

$$\sup_{f \in \mathcal{F}_W} \left| \mathbb{E}\left[\sigma^2 f'(F_T) - F_T f(F_T) \right] \right|,$$

where $F_T = M_T/\sqrt{T}$ and $\sigma^2 = \frac{\mathbb{E}[X^2]\mu}{1-\|\phi\|_1}$.

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- We have $F_T=\delta\left(\left(\frac{x\mathbb{1}_{s\leq T}}{\sqrt{T}}\right)_{(s,x)\in\mathbb{R}_+\times\mathbb{R}}\mathcal{Z}\right)$ which yields thanks to the duality formula

$$\begin{aligned}\mathbb{E}\left[F_Tf(F_T)\right]&=\mathbb{E}\left[\delta\left(\left(\frac{x\mathbb{1}_{s\leq T}}{\sqrt{T}}\right)_{(s,x)}\mathcal{Z}\right)f(F_T)\right]\\&=\mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}}\frac{x\mathbb{1}_{s\leq T}}{\sqrt{T}}D_{(s,x)}f(F_T)\lambda_s\mathrm{d}s\nu(\mathrm{d}x)\right]\\&=\mathbb{E}\left[\frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}}xD_{(s,x)}f(F_T)\lambda_s\mathrm{d}s\nu(\mathrm{d}x)\right].\end{aligned}$$

Expansions

- A Taylor expansion yields

$$D_{(s,x)}f(F_T) = f'(F_T)D_{(s,x)}F_T + \frac{1}{2}f''(\bar{F})|D_{(s,x)}F_T|^2.$$

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- Hence

$$\begin{aligned}\mathbb{E}\left[\sigma^2 f'(F_T) - F_T f(F_T)\right] =& \mathbb{E}\left[\sigma^2 f'(F_T)\right] \\ & - \frac{1}{\sqrt{T}}\mathbb{E}\left[f'(F_T)\int_0^T \int_{\mathbb{R}} x D_{(s,x)}F_T \lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right] \\ & - \frac{1}{2\sqrt{T}}\mathbb{E}\left[f''(\bar{F})\int_0^T \int_{\mathbb{R}} x|D_s F_T|^2 \lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right].\end{aligned}$$

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- With $D_{(s,x)}F_T = \frac{1}{\sqrt{T}}\left(x + \hat{M}_T^{(s,x)}\right)$.

Reckless bound (1)

It is possible to directly factor and use $\|f'\|_\infty \leq 1$

$$\left|\mathbb{E}\left[f'(F_T)\left(\sigma^2 - \frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} xD_{(s,x)}F_T\lambda_s\mathrm{d}s\nu(\mathrm{d}x)\right)\right]\right|$$
$$\leq \mathbb{E}\left[\left|\sigma^2 - \frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} xD_{(s,x)}F_T\lambda_s\mathrm{d}s\nu(\mathrm{d}x)\right|\right].$$

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This bound can in turn be separated in three terms

1. $A_{1,1} = \left|\sigma^2 - \frac{\mathbb{E}[X^2]}{T}\int_0^T \mathbb{E}[\lambda_s]\mathrm{d}s\right| = O\left(\frac{1}{T}\right),$
2. $A_{1,2} = \frac{\mathbb{E}[X^2]}{T}\mathbb{E}\left[\left|\int_0^T \lambda_s - \mathbb{E}[\lambda_s]\mathrm{d}s\right|\right] = O\left(\frac{1}{\sqrt{T}}\right),$

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3. $A_{1,3} = \frac{|\mathbb{E}[X]|}{T}\mathbb{E}\left[\left|\int_0^T\int_{\mathbb{R}} x\lambda_s\hat{M}_T^{(t,x)}\mathrm{d}s\nu(\mathrm{d}x)\right|\right]$.

Reckless bound (2)

- Recall that $F_T = \frac{M_T}{\sqrt{T}} = \frac{S_T - \mathbb{E}[X]\int_0^T \lambda_s \mathrm{d}s}{\sqrt{T}}$.
- The Gaussian limit is centered and its variance is $\sigma^2 = \frac{\mathbb{E}[X^2]\mu}{1-\|\phi\|_1}$.

Theorem

If $\int_0^{+\infty} s\phi(s)\mathrm{d}s < +\infty$ and $\mathbb{E}[X^2] < +\infty$ then there exists a constant $C_{\mu,\phi,\nu}$ independent from T such that

$$d_W(F_T, G) \leq \frac{C_{\mu,\phi,\nu}}{\sqrt{T}} \left(1 + \frac{1}{\sqrt{T}} \mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}} x\lambda_s \hat{M}_T^{(s,x)} \mathrm{d}s\nu(\mathrm{d}x)\right|\right]\right),$$

where $G \sim \mathcal{N}(0, \sigma^2)$.

Commutation property (1)

- Instead, we make the following observation

$$D_{(s,x)}F_T = D_{(s,x)}\delta\left(\left(x\frac{\mathbb{1}_{s\leq T}}{\sqrt{T}}\right)_{(s,x)\in\mathbb{R}_+\times\mathbb{R}}\mathcal{Z}\right)$$

and benefit from the following commutation property (for a deterministic z)

$$D_{(s,x)}\delta(z\mathcal{Z}) = z_{(s,x)} + \delta(z\hat{\mathcal{Z}}^s)$$

where $\hat{\mathcal{Z}}^s_{(r,\theta)} = \mathbb{1}_{r>s}\mathbb{1}_{\lambda_r<\theta\leq\lambda_r\circ\varepsilon_{(s,1)}}$.

Commutation property (2)

Using this property we have that

$$\mathbb{E}\left[f'(F_T)\left(\sigma^2 - \frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} xD_{(s,x)}F_T\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right]$$

$$=\mathbb{E}\left[f'(F_T)\left(\sigma^2 - \frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} \frac{x^2}{\sqrt{T}}\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right]$$

$$+\mathbb{E}\left[\frac{f'(F_T)}{\sqrt{T}}\left(\int_0^T\int_{\mathbb{R}} \delta(z\hat{\mathcal{Z}}^s)\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right].$$

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$$=\mathbb{E}\left[f'(F_T)\left(\sigma^2 - \frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} \frac{x^2}{\sqrt{T}}\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right] + \mathbb{E}\left[\frac{f'(F_T)}{\sqrt{T}}\left(\int_0^T\int_{\mathbb{R}} \delta(z\hat{\mathcal{Z}}^s)\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right].$$

And we show that

$$\mathbb{E}\left[f'(F_T)\left(\int_0^T\int_{\mathbb{R}} \delta(z\hat{\mathcal{Z}}^s)\lambda_s \mathrm{d}s\nu(\mathrm{d}x)\right)\right] = \mathbb{E}\left[\int_0^T\int_{\mathbb{R}} \lambda_s\mathbb{E}_s[f'(F_T)\delta(z\hat{\mathcal{Z}}^s)]\mathrm{d}s\nu(\mathrm{d}x)\right] = 0.$$

Application to the normalized martingale

- Recall that $F_T = \frac{M_T}{\sqrt{T}} = \frac{S_T - \mathbb{E}[X] \int_0^T \lambda_s \mathrm{d}s}{\sqrt{T}}$.
- The Gaussian limit is centered and its variance is $\sigma^2 = \frac{\mathbb{E}[X^2]\mu}{1 - \|\phi\|_1}$.

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Theorem

If $\int_0^{+\infty} s\phi(s)\mathrm{d}s < +\infty$ and $\mathbb{E}[X^2] < +\infty$, then there exists $C_{\mu,\phi,\nu} > 0$ (depending only on μ, ϕ, ν) such that

$$d_W(F_T, G) \leq \frac{C_{\mu,\phi,\nu}}{\sqrt{T}},$$

where $G \sim \mathcal{N}(0, \sigma^2)$.

Generalisation (deterministic compensator)

Theorem

Assume that $\mathbb{E}[X^2] < +\infty$ and let

$$\varpi := \mu \frac{\mathbb{E}[X]}{1 - \|\phi\|_1} \quad \text{and} \quad \Gamma_T := \frac{S_T - \varpi T}{\sqrt{T}}, \quad T > 0.$$

Then, there exists $C'_{\mu,\phi,\nu} > 0$ (depending only on $\mu, \|\phi\|_1, \nu$) such that

$$d_W(\Gamma_T, \mathcal{N}(0, \zeta^2)) \leq \frac{C'_{\mu,\phi,\nu}}{\sqrt{T}}, \quad T > 0,$$

where

$$\zeta^2 := \mu \frac{\mathbb{E}[X^2] + \|\phi\|_1(\mathbb{E}[X^2] - (\mathbb{E}[X_1])^2)(\|\phi\|_1 - 2)}{(1 - \|\phi\|_1)^3}.$$

Merci!

List of articles

- Normal approximation of compound Hawkes functionals.
  With N. Privault and A. Réveillac. Published in Journal of Theoretical Probability
- The Malliavin-Stein method for Hawkes functionals.
  With C. Hillairet, L. Huang and A. Réveillac. Published in ALEA.
- The Malliavin-Stein method for the multivariate compound Hawkes process.
  Submitted.