Non co-adapted Couplings of Brownian motion in subRiemannian manifolds

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SubRiemannian structure of the Heisenberg group

Example (The Heisenberg group)

- $\mathbb{H} = (\mathbb{R}^3, \star)$ *such that for* $(x,y,z),\ (x',y',z') \in \mathbb{H}$,

$$(x,y,z) \star (x',y',z') = (x+x', y+y', z+z' + \frac{1}{2}(xy' - yx'))$$

- ***Horizontal space :*** $\mathcal{H}_{(x,y,z)} = Vect(X_{(x,y,z)}, Y_{(x,y,z)})$
  with $X_{(x,y,z)} = \partial_x - \frac{y}{2}\partial_z$ *and* $Y_{(x,y,z)} = \partial_y + \frac{x}{2}\partial_z$
- ***Horizontal curve :*** $\gamma : I \to \mathbb{H}$ *smooth such that,* $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}\ \forall\ t \in I$
- ***Carnot-Carathéodory distance between*** g ***and*** $h \in \mathbb{H}$ *:*

$$d_{cc}(g,h) = \inf\{L(\gamma) \mid \gamma \textit{ horizontal curve between } g \textit{ et } h\}$$

$$\textit{with } L(\gamma) = \int_I \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\mathcal{H}_{\gamma(t)}}}\,dt = \int_I ||\dot\gamma_{\mathbb{R}^2}(t)||_{\mathbb{R}^2}\,dt$$

- *On a en particulier :* $d_{cc}(0,(x,y,z))^2 \sim ||(x,y)||^2 + |z|$

Brownian motion on subRiemannian manifolds

Definition of the Brownian motion on the Heisenberg group

Definition

- *We define the Sub-Laplacian operator by* $L = \frac{1}{2}(X^2 + Y^2)$ *;*
- *The Brownian motion on a subRiemannian manifold is defined as the diffusion process of infinitesimal generator* L*.*

- The Brownian motion starting at $(x_1, x_2, z) \in \mathbb{H}$ is given by :

$$\mathbb{B}_t := \left(B_t^1, B_t^2, z + \frac{1}{2}\left(\int_0^t B_s^1 dB_s^2 - \int_0^t B_s^2 dB_s^1 \right) \right)$$

with $(B_t^1, B_t^2)_t$ a Brownian motion on $\mathbb{R}^2$ starting at (x_1, x_2).

- Taking polar coordinates (φ_t, θ_t) for (B_t^1, B_t^2), $\mathbb{B}_t$ satisfies : $\begin{cases} d\varphi_t = dC_t^1 + \frac{1}{\varphi_t} dt \\ d\theta_t = \frac{1}{\varphi_t} dC_t^2 \\ dz_t = \frac{\varphi_t}{2} dC_t^2 \end{cases}$ with

C_t^1, C_t^2 two real independent Brownian motions.

Brownian motion on subRiemannian manifolds

Generalisation to $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$

We consider two other subRiemannian manifolds that we will generally denote G and we associate a real k to each :

- $SU(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) \mid A \text{ unitary and } det(A) = 1\}$, $k = 1$;
- $SL(2,\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid det(A) = 1\}$, $k = -1$.

Using cylindrical coordinates, the Brownian motion on G satifies :

$$\begin{cases} d\varphi_t = dB_t^1 + \sqrt{k} cotan(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{sin(\sqrt{k}\varphi_t)} dB_t^2 \\ dz_t = \frac{1}{\sqrt{k}} tan(\frac{\sqrt{k}\varphi_t}{2}) dB_t^2 \end{cases}$$

with B_t^1, B_t^2 two real independent Brownian motions.

Geometrical interpretation :

- $X_t = (\varphi_t, \theta_t)$ is a Brownian motion in spherical/polar coordinates on a Riemannian manifold M of curvature k ;
- z_t is the area swept by $(X_s)_{s\leq t}$ on M relative to the chosen pole **modulo** 4π.

Successful couplings of Brownian motion

Example of couplings

Definition

Constructing a coupling of Brownian motions starting at (x, x') *consists in studying the joint law of* $(\mathbb{B}_t, \mathbb{B}'_t)_t$ *with* $\mathbb{B}_t$ *a Brownian motion starting at* x *and* $\mathbb{B}'_t$ *a Brownian motion starting at* x'.

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On G, we need to couple :

$$\begin{cases} d\varphi_t = dB_t^1 + \sqrt{k} cotan(\sqrt{k}\varphi_t)dt \\ d\theta_t = \frac{\sqrt{k}}{sin(\sqrt{k}\varphi_t)} dB_t^2 \\ dz_t = \frac{1}{\sqrt{k}} tan(\frac{\sqrt{k}\varphi_t}{2}) dB_t^2 \end{cases} \quad \text{and} \quad \begin{cases} d\varphi'_t = dB'^1_t + \sqrt{k} cotan(\sqrt{k}\varphi'_t)dt \\ d\theta'_t = \frac{\sqrt{k}}{sin(\sqrt{k}\varphi_t)} dB'^2_t \\ dz'_t = \frac{1}{\sqrt{k}} tan(\frac{\sqrt{k}\varphi_t}{2}) dB'^2_t \end{cases} .$$

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Example

$B^1_t = -B^{1\prime}_t$ *and* $B^2_t = B^{2\prime}_t$.

Remarque

For $k = 1$, if $\varphi_0 = \pi - \varphi'_0$ and $\theta_0 = \theta'_0$ we get $\varphi_t = \pi - \varphi'_t$ for all $t \geq 0$. We obtain a reflection coupling.

Successful couplings of Brownian motion

Successful couplings

Definition

We considere a coupling $(\mathbb{B}_t, \mathbb{B}'_t)$ *starting at* (x, x'). *We define* $\tau := inf\{t > 0 | \mathbb{B}_t = \mathbb{B}'_t\}$. *If* $\tau < +\infty$ *a.s., our coupling is called successful.*

Motivations for successful couplings :

- To obtain estimations of the total variation distance :

$$\textbf{Aldous inequality : } d_{TV}(\mathcal{L}(\mathbb{B}_t), \mathcal{L}(\mathbb{B}'_t)) \leq \mathbb{P}(t < \tau)$$

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$$|P_t f(\mathbb{B}_x) - P_t f(x')| \leq 2||f||_\infty \mathbb{P}(t < \tau)$$

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- To study harmonic functions :

Theorem (Wang, 2002)

On a Riemannian manifold with Ricci curvature bounded below, there exists a successful coupling of Brownian motions if and only if all harmonic bounded function are constant.

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Example

There is no successful coupling on the hyperbolic plane.

Successful coupling on subRiemannian manifolds

Problem

Can we define a successful coupling $((X_t, z_t), (X'_t, z'_t))$ on G ?

- **On $\mathbb{H}$?** Yes : Kendall (2010), **co-adapted coupling** ; Banerjee, Gordina, Mariano (2017) **non co-adapted coupling** ;
- **On $SU(2, \mathbb{C})$?** Yes : B. (2023), **co-adapted coupling and non codapted coupling**
- **On $SL(2, \mathbb{R})$?** No : there is no successful coupling on the hyperbolic space.

Remarque

Co-adapted couplings are easy to simulate but, in general, for these couplings, $\mathbb{P}(t < \tau)$ is not optimal and difficult to compute.

Coupling rates on $\mathbb{H}$ under the hypothesis $X_0 = X'_0$; *for t large enough, we have :*

Kendall co-adapted coupling :

$$\mathbb{P}(t < \tau) \geq \frac{C}{\sqrt{t}}$$

Banerjee et. al. non co-adapted coupling :

$$\mathbb{P}(t < \tau) \leq \frac{C'}{t})$$

Successful coupling on subRiemannian manifolds

Brownian bridges coupling

Theorem

Let suppose $X_0 = X_0'$. There exists a non co-adapted coupling on $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$ and There exists $c_1, c_2 > 0$ such that : $\mathbb{P}(t < \tau) \leq c_1 e^{-c_2 t}$ for t large enough.

Elements of proof

This proof is inspired by the coupling on $\mathbb{H}$ from Banerjee et. al. We choose $T > 0$. We first define a coupling on $[0, T]$:

- $B_t^1 = {B_t^1}' \Rightarrow \varphi_t = \varphi_t'$ *for all* t *;*

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Elements of proof

This proof is inspired by the coupling on $\mathbb{H}$ *from Banerjee et. al. We choose* $T > 0$. *We first define a coupling on* $[0, T]$ *:*

- $B_t^1 = B_t^{1\prime} \Rightarrow \varphi_t = \varphi_t'$ *for all* t *;*
- $\theta_t = \theta_0 + C_{\sigma(t)}$, $\theta_t' = \theta_0 + C'_{\sigma(t)}$ *with* C *and* C' *two real Brownian motions and* $\sigma(t) = \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi_s)} ds$ *;*

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- $B_t^1 = {B_t^1}' \Rightarrow \varphi_t = \varphi_t'$ *for all* t *;*
- $\theta_t = \theta_0 + C_{\sigma(t)}$, $\theta_t' = \theta_0 + C'_{\sigma(t)}$ *with* C *and* C' *two real Brownian motions and* $\sigma(t) = \int_0^t \frac{k}{\sin^2(\sqrt{k}\varphi_s)} ds$ *;*
- $\begin{cases} \theta_t = B^{br}_{\sigma(t)} + \frac{\sigma(t)}{\sigma(T)} G \\ \theta_t' = B^{br'}_{\sigma(t)} + \frac{\sigma(t)}{\sigma(T)} G \end{cases}$ *with* B^{br} *and* $B^{br'}$ *two Brownian bridges on* $[0, \sigma(T)]$ *and* $G \sim \mathcal{N}(0, \sigma(T))$, *independent to the Brownian bridges.*

Remarque

We get $X_T = X_T'$.

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Theorem

There exists $c_1, c_2 > 0$ such that, if $X_0 = X'_0$, there exists a non co-adapted coupling on $SU(2, \mathbb{C})$ and $SL(2, \mathbb{R})$ satisfying : $\mathbb{P}(t < \tau) \leq c_1 e^{-c_2 t}$.

- Decomposition of the Brownian bridges using Karhunen-Loève formula :

$$B^{br}_{\sigma(t)} = \sqrt{\sigma(T)} \sum_{j \geq 1} Z_j \frac{\sqrt{2}}{j\pi} \sin(\frac{j\pi\sigma(t)}{\sigma(T)}) \text{ and } B^{br'}_{\sigma(t)} = \sqrt{\sigma(T)} \sum_{j \geq 1} Z'_j \frac{\sqrt{2}}{j\pi} \sin(\frac{j\pi\sigma(t)}{\sigma(T)})$$

with $(Z_j)_j$ (resp. $(Z'_j)_j$) a sequence of independent standard Gaussian variables, independent of B^1.

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  with $(Z_j)_j$ (resp. $(Z_j')_j$) a sequence of independent standard Gaussian variables, independent of B^1.

- We choose $Z_j = Z_j'$ except for $j = 1$.

  We obtain : $\theta_t - \theta_t' = (Z_1 - Z_1')\frac{\sqrt{2\sigma(T)}}{\pi} \sin(\frac{\pi\sigma(t)}{\sigma(T)})$.

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- We have :
  $z_T - z_T' = z_0 - z_0' + K(T)\frac{Z_1 - Z_1'}{2}$ with $K(T) = 2\sqrt{\frac{2}{\sigma(T)}} \int_0^T \frac{1}{1+\cos(\varphi_s)} \cos(\frac{\pi\sigma(s)}{\sigma(T)}) ds$.
  We want : $\mathbb{P}\left(z_T - z_T' \equiv 0(4\pi)\right) \neq 0$.

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  We want : $\mathbb{P}\Big(z_T - z_T' \equiv 0(4\pi)\Big) \neq 0$.
- We take $(W_t)_t$ a Brownian motion independent of B^1, G and $(Z_j)_{j\geq 2}$. We define
  $\omega = \inf\{t | W_t \notin] - \frac{z_0 - z_0'}{K(T)}, \frac{-(z_0-z_0')+4\pi}{K(T)}[\}$. Then we define $W_t' := \begin{cases} -W_t & \text{if } t \leq \omega \\ W_t - 2W_\omega & \text{else} \end{cases}$.
  We choose : $Z_1 := -W_1 \sim \mathcal{N}(0,1)$ and $Z_1' := W_1' \sim \mathcal{N}(0,1)$.

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  We choose : $Z_1 := -W_1 \sim \mathcal{N}(0,1)$ and $Z_1' := W_1' \sim \mathcal{N}(0,1)$. With this coupling strategy we have two cases :
  - If $\omega \leq 1$, then $K(T)\frac{Z_1 - Z_1'}{2} = K(T)W_\omega \equiv -(z_0 - z_0')(4\pi)$.
  - If $\omega > 1$, then $K(T)\frac{Z_1 - Z_1'}{2} = K(T)W_1 \not\equiv -(z_0 - z_0') \mod (4\pi)$.

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There exists $c_1, c_2 > 0$ such that, if $X_0 = X_0'$, there exists a non co-adapted coupling on $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$ satisfying : $\mathbb{P}(t < \tau) \leq c_1 e^{-c_2 t}$.

- We have :
  $z_T - z_T' = z_0 - z_0' + K(T)\frac{Z_1 - Z_1'}{2}$ with $K(T) = 2\sqrt{\frac{2}{\sigma(T)}}\int_0^T \frac{1}{1+\cos(\varphi_s)}\cos(\frac{\pi\sigma(s)}{\sigma(T)})ds$.
  We want : $\mathbb{P}\Big(z_T - z_T' \equiv 0(4\pi)\Big) \neq 0$.
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- We reproduce this coupling until we get $\omega > 1$.

Some references :

- Coupling in the Heisenberg group and its applications to gradient estimates,S. Banerjee, M. Gordina, P Mariano, 2017
- The subelliptic heat kernel on $SU(2)$: representations, asymptotics and gradient bounds, F.Baudoin, M.Bonnefont, 2009
- Couplings of Brownian motions on $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$, M.Bénéfice, 2023, Preprint
- Coupling time distribution asymptotics for some couplings of the Levy stochastic area, W. S. Kendall, 2010
- Liouville theorem and coupling on negatively curved Riemannian manifolds, F.-Y. Wang, 2002

Thank you for your attention.