Coupling of Brownian motions with set valued dual processes on Riemannian manifolds

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joint work with

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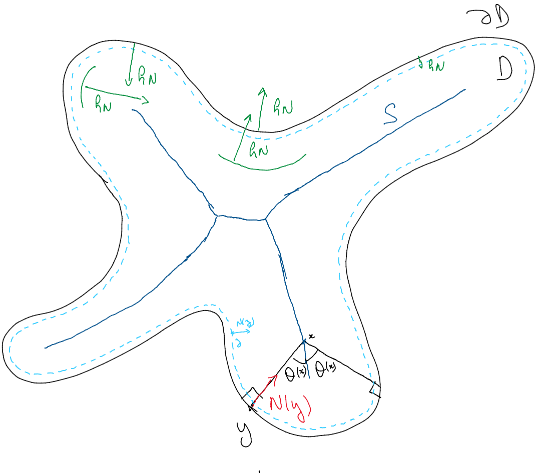
1. Stochastic renormalized mean curvature flow and intertwined Brownian motion

2. Flows in $\mathbb{R}^2$ and lifetimes

Domain and skeleton



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First we investigate the case of n-dimensional spheres $S^n \subset \mathbb{R}^{n+1}$.

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Consequence: the same asymptotic holds for the strong stationary time $\tau(n)$ for the Brownian motion X_t started at N ($X_{\tau(n)}$ is uniformly distributed in S^n).

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Link to Bessel(3) process : Up to the stopping time until which everything is defined, we always have:

Theorem ([Coulibaly-Miclo:18])

The volume process $\left(\mu(\tilde{D}_{\tau(t)})\right)_{t\geq 0}$ is a Bessel process of dimension 3, where the time change $\tau(t)$is the inverse of

$$t \mapsto \int_0^t \left(\underline{\mu}(\partial\tilde{D}_s)\right)^2 ds$$

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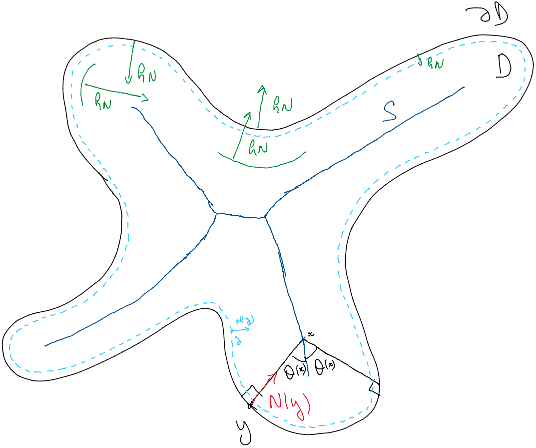

A general coupling equation

- For each closed domain $D \subset M$ with smooth boundary, let $f^D : D \to \mathbb{R}$ satisfy: $\|\nabla f^D\|_\infty \leq 1$, $\nabla \mathbf{f^D} = \nabla \rho_{\partial \mathbf{D}}$ **around boundary**, $(x, D) \mapsto f^D(x)$ is sufficiently regular.

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This is the equation for the renormalized stochastic mean curvature flow.

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- $\mathcal{L}$ the generator of (X_t, D_t) stopped before hitting time by D_t of its inner and outer skeleton.
- $\tilde{\mathcal{L}}$ the generator of $\tilde{D}_t$ with same stopping time, and $\tilde{\nu}_t$ its law at time t
- $\nu_t(dx, dD) = \tilde{\nu}_t(dD)\mathcal{U}^D(dx)$

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(1) $\partial_t \nu_t(\cdot) = \nu_t(\mathcal{L}(\cdot))$. As a consequence ν_t is invariant by the semigroup $\mathcal{P}_t$ of (X_t, D_t),

(2) For all D-stopping time τ, conditional to $D_{[0,\tau]}$, X_τ has uniform law in D_τ.

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Remark

- A key ingredient for the proof is Stokes theorem

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with h^{D_t} mean curvature of level sets of $\rho_{\partial D_t}$,
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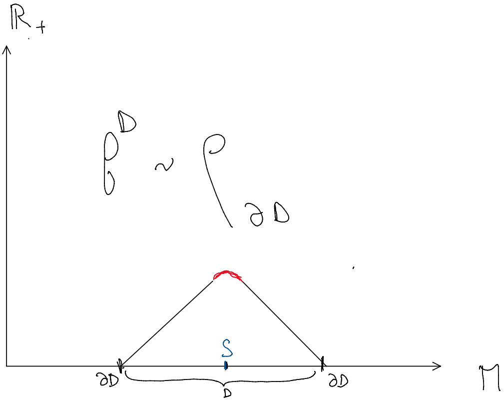
Remark

- (2), (3) is a generalization of Pitman $2M - X$ theorem

Full coupling



Full decouplings and local times on boundary

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The equations for X_t and D_t are

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where W_t is a real valued Brownian motion and $L_t^{\partial D_\cdot}(X)$ is the local time of $X_\cdot$ on $\partial D_\cdot$.
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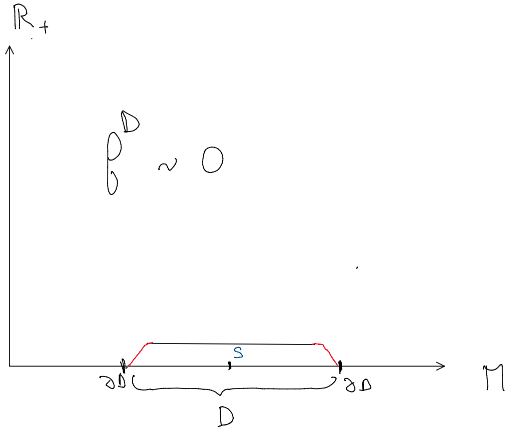
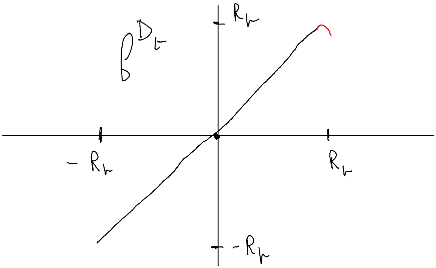


Pitman coupling



$$-R_r = X_r - 2M_r$$

Mean curvature flow in $\mathbb{R}^2$

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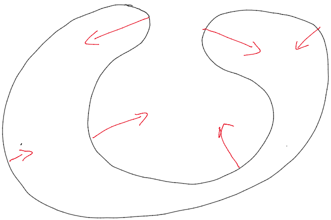
- The solution becomes more and more round (isoperimetric ratio converges to 1) and shrinks to a point in finite time (Huisken 84, Gage 83, Gage 84, Gage-Hamilton 86).



Renormalized mean curvature flow in $\mathbb{R}^2$

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  - finally we prove infinite lifetime and convergence to a disk.

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- Small time existence for $C^{2+\alpha}$ initial curve ($\alpha > 0$) (Coulibaly-Miclo 2018);
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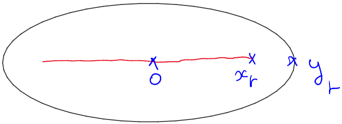
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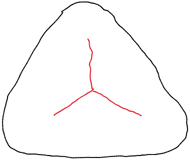

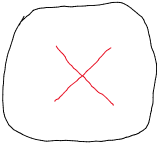

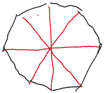

Thanks for your attention