La limite super-diffusive de la marche aléatoire de l'éléphant

Journées de Probabilités, Angers

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avec Hélène Guérin et Kilian Raschel

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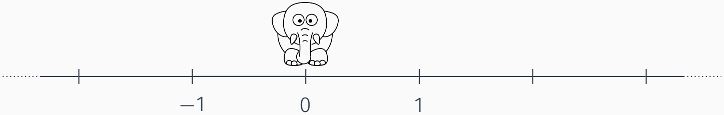

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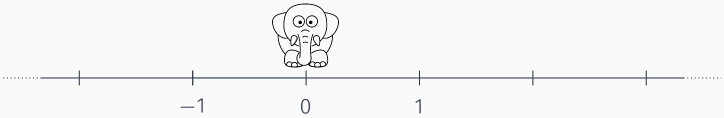
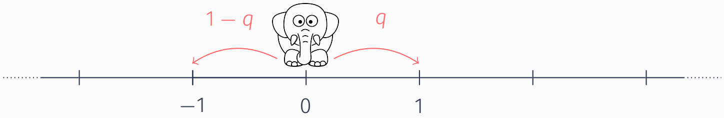


A time $n = 1$



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$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad 1-p. \end{cases}$$

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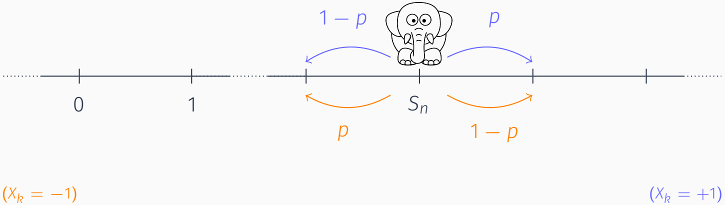

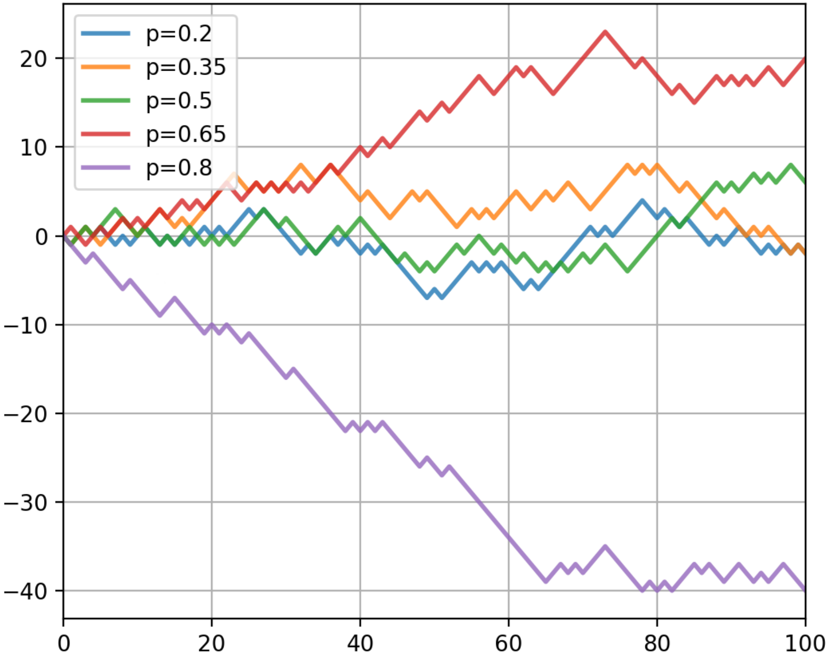
p=0.2
p=0.35
p=0.5
p=0.65
p=0.8
20
10
0
−10
−20
−30
−40
0
20
40
60
80
100

A martingale approach

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then,

$$\begin{aligned}\mathbb{P}(X_{n+1} = 1 \mid \mathcal{F}_n) &= p\frac{\#\{\text{steps to the right}\}}{n} + (1-p)\frac{\#\{\text{steps to the left}\}}{n} \\ &= p\frac{S_n + n}{2n} + (1-p)\frac{n - S_n}{2n} \\ &= \frac{1}{2}\left(1 + (2p-1)\frac{S_n}{n}\right).\end{aligned}$$

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The conditionnal distribution of X_{n+1} given the past is

$$\mathcal{L}(X_{n+1} \mid \mathcal{F}_n) = \mathcal{R}(p_n)$$

where $p_n = \frac{1}{2}\Big(1 + a\frac{S_n}{n}\Big)$ and $a = 2p - 1$.

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We deduce that

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = S_n + (2p_n - 1) = \left(1 + \frac{a}{n}\right) S_n = \gamma_n S_n.$$

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Consequently, we set

$$M_n = a_n S_n$$

where $a_1 = 1$ and

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)}.$$

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The process (M_n) is a locally bounded square-integrable martingale. Indeed,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = a_{n+1}\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = a_{n+1}\gamma_n S_n = a_n S_n = M_n$$

and $\mathbb{E}[M_n^2] \leq (n a_n)^2$.

Three regimes

It is possible tho show that

$$\langle M \rangle_n = \sum_{k=1}^{n} a_k^2 - a^2 \sum_{k=1}^{n} a_k^2 \left(\frac{S_k}{k} \right)^2.$$

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- the diffusive regime where $a < 1/2$ and $v_n = O(n^{1-2a})$,
- the critical regime where $a = 1/2$ and $v_n = O(\log n)$,
- the superdiffusive regime where $a > 1/2$ and $v_n = O(1)$.

Main results

Baur and Bertoin 2016, Coletti et al. 2017, Bercu 2018, Kubota and Takei 2019...

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Theorem (Law of large numbers)

Diffusive

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{a.s.}{=} 0$$

Critical

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} \overset{a.s.}{=} 0$$

Superdiffusive

$$\lim_{n\to\infty} \frac{S_n}{n^a} \overset{a.s./\mathbb{L}^m}{=} L_q$$

Theorem (Asymptotic normality)

Diffusive

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2a}\right)$$

Critical

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

Superdiffusive

$$\frac{S_n - n^a L_q}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2a-1}\right)$$

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Theorem

Diffusive

$$\left(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}},\ t\geq 0\right) \underset{n\to\infty}{\Longrightarrow} (W_t,\ t\geq 0)$$

where W_t is a centered gaussian process

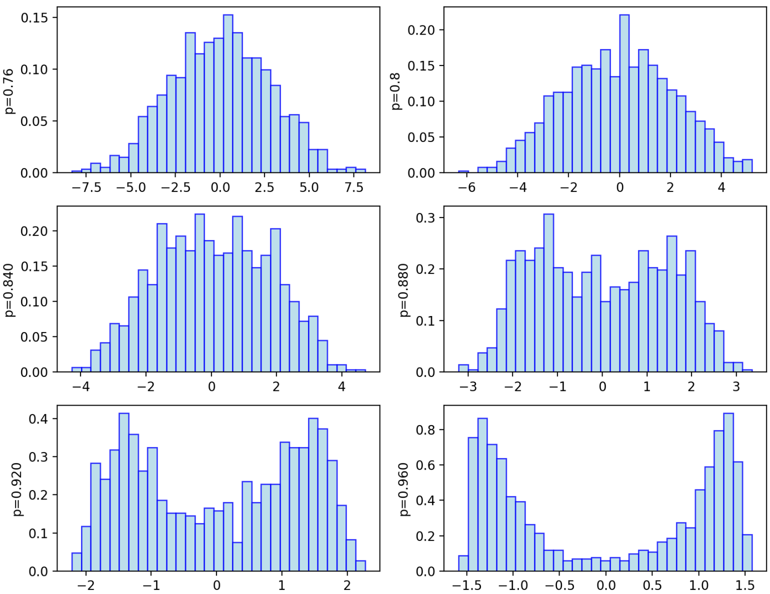
$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} t^a s^{1-a}, \quad 0 < s \leq t$$

Critical

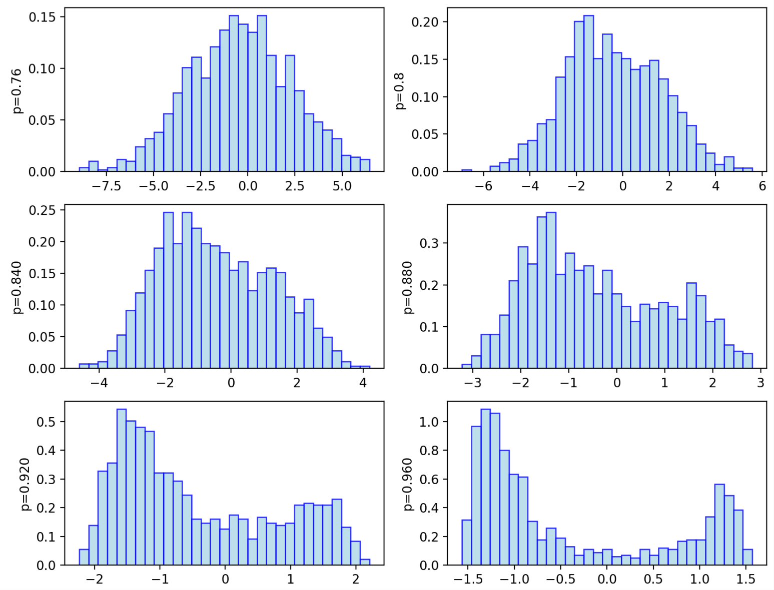
$$\left(\frac{S_{\lfloor n^t\rfloor}}{\sqrt{n^t\log n}},\ t\geq 0\right) \underset{n\to\infty}{\Longrightarrow} (B_t,\ t\geq 0)$$

Superdiffusive

$$\left(\frac{S_{\lfloor nt\rfloor}}{n^a},\ t\geq 0\right) \underset{n\to\infty}{\Longrightarrow} (t^a L,\ t\geq 0)$$



Histogram of L when $q = 0.5$



Histogram of L when $q = 0.3$

New insights on L

Thanks to the connection with random recursive trees on which a Bernoulli bond percolation hase been performed, it holds that :

The distribution of L

$$L = \sum_{i=1}^{\infty} C_i \cdot Z_i = C_1 \cdot \sum_{i=1}^{\infty} (\beta_{\tau_i})^a \cdot Z_i.$$

such that (Z_i) are i.i.d. $\mathcal{R}(1/2)$, C_1 has a Mittag-Leffler distribution with parameter a and C_i a random variable with the same law as $(\beta_{\tau_i})^a \cdot C_1$, where β_i denotes a beta variable with parameter $(1, i-1)$ and is further independent of C_1.

An elephant inside an urn ?

A few references

S. Janson – Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Processes and their Applications* (2004)

B. Chauvin, C. Mailler, N. Pouyanne – Smoothing equations for large Pólya urns. *J. Theoret. Probab* (2015)

E. Baur and J. Bertoin – Elephant random walks and their connection to Pólya-type urns. *Physical review E* (2016)

The ERW and the associated Pólya urn

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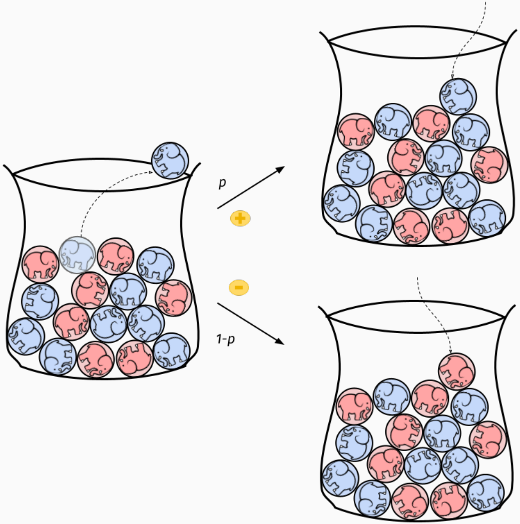

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1 (red ball)	going right
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such that

$$A = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = 2p - 1 = a, \quad v_1 = \binom{1}{1},\ v_2 = \binom{1}{-1}.$$

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In this case, S_n has the same distribution as $R_n - B_n = 2R_n - n$.

A few results on the ERW-Pólya urn

Theorem (Janson, 2004)

When $a > 1/2$ and $U(0) = (\alpha,\ \beta)^T$, it is true that

$$\lim_{n\to\infty} \frac{U(n) - nv_1}{n^a} = W_{(\alpha,\beta)} v_2 \quad a.s.$$

where $W_{(\alpha,\beta)}$ is a non-degenerate random variable such that $\mathbb{E}[W_{(\alpha,\beta)}] = \dfrac{\alpha - \beta}{\Gamma(1+a)}$ and $\mathbb{E}[W^2_{(\alpha,\beta)}] = \dfrac{1}{(2a-1)\Gamma(2a)}$. In particular,

$$\lim_{n\to\infty} \frac{R_n - B_n}{n^a} = W_{(\alpha,\beta)} \quad a.s.$$

Moreover, we can show that $W_{(1,0)} \overset{a.s.}{=} -W_{(0,1)}$. Hence, it follows that

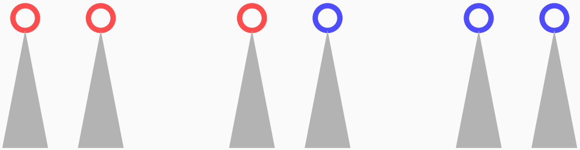
$$L_q \overset{\mathcal{L}}{=} Z_q W$$

where $Z_q \sim \mathcal{R}(q)$ is independent of $W := W_{(1,0)}$.

Urns and trees

with probability q 🔴 🔵 with probability $1 - q$

Urns and trees



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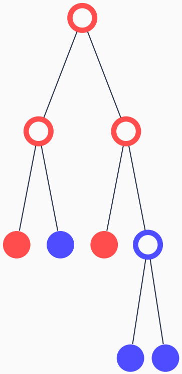
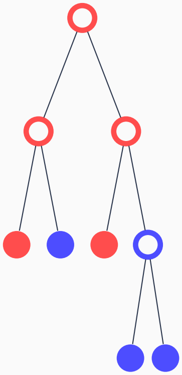


At time n,

- $N_k(n)$ is the number of leaves of the k-th subtree,
- the number of drawings in the k-th subtree is $N_k(n) - 1$ (time inside).

Urns and trees



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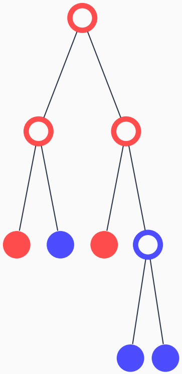

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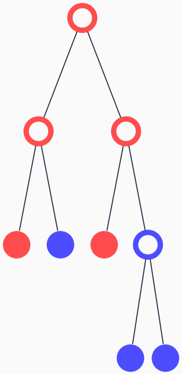

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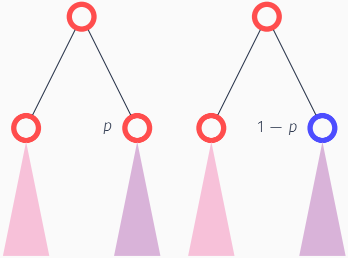
Remember that the balls of the whole urn are uniformly drawn at any time and notice that

- at each drawing in the k-th subtree, $N_k(n)$ increases by 1
- $N(n) = (N_1(n), N_2(n)))$ has exactly the same distribution as the 2-color Pólya urn process having I_2 as (deterministic) replacement matrix and $(1, 1)$ as initial composition.

Distributional equation (1)



Consider simultaneously

- an urn process $N = (N_1, N_2)$ having I_2 as replacement matrix and $(1, 1)$ as initial condition,
- two urn processes $U^{(1)}_{(1,0)}$ and $U^{(2)}_{(1,0)}$ having A as mean replacement matrix and $(1, 0)$ as initial condition,
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all these processes being independent of each other.

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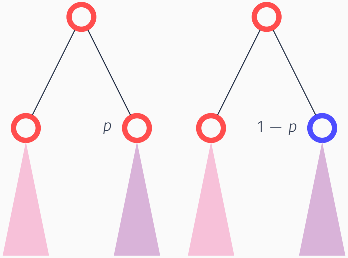

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Then, the process $U_{(1,0)} = (U_{(1,0)}(n))_n$ has the same distribution as

$$U_{(1,0)}(n) \overset{\mathcal{L}}{=} U^{(1)}_{(1,0)}(N_1(n) - 1)$$

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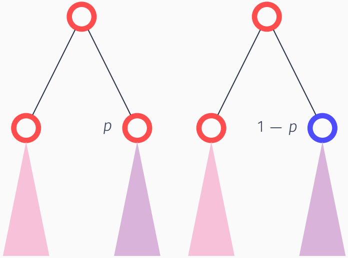

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$$U_{(1,0)}(n) \overset{\mathcal{L}}{=} U^{(1)}_{(1,0)}\big(N_1(n) - 1\big) + \xi_p U^{(2)}_{(1,0)}\big(N_2(n) - 1\big)$$

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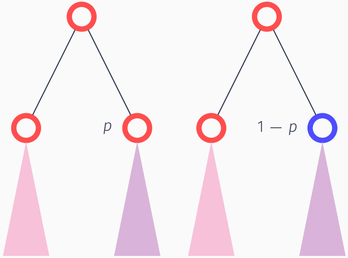

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It is known that

$$\lim_{n\to\infty} \frac{1}{n} N_n = (V, 1 - V) \quad \text{a.s.} \quad \text{where } V \sim \mathcal{U}(0,1)$$

Distributional equation (2)

$$U_{(1,0)}(n) \overset{\mathcal{L}}{=} U^{(1)}_{(1,0)}\big(N_1(n) - 1\big) + \xi_p U^{(2)}_{(1,0)}\big(N_2(n) - 1\big) + (1 - \xi_p) U^{(2)}_{(0,1)}\big(N_2(n) - 1\big)$$

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which leads to

$$\begin{aligned} W_{(1,0)} &\overset{\mathcal{L}}{=} V^a W^{(1)}_{(1,0)} + \xi_p (1 - V)^a W^{(2)}_{(1,0)} + (1 - \xi_p)(1 - V)^a W^{(2)}_{(0,1)} \\ &\overset{\mathcal{L}}{=} V^a W^{(1)}_{(1,0)} + (2\xi_p - 1)(1 - V)^a W^{(2)}_{(1,0)}. \end{aligned}$$

Distributional equation (2)

Theorem (Guérin, L., Raschel – 2023+)

Let W_{v_2} be the limit of a large two-color Pólya urn process with random replacement matrix A, initial composition $(0, 1)$ and ratio $a > 1/2$. Then,

$$W \overset{\mathcal{L}}{=} V^a W^{(1)} + Z_p (1 - V)^a W^{(2)}$$

where

- *V is a uniformly distributed random variable on $[0, 1]$,*
- *Z_p is a Rademacher distributed random variable with parameter p,*
- *the $W^{(k)}$ are copies of W, all being independent of each other and of V and Z_p.*

The superdiffusive limit (1)

Recall that $L_q \stackrel{\mathcal{L}}{=} Z_q W$.

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Theorem (2023+)

Let $(m_k)_{k\geq 1}$ be defined by $m_1 = 1$ and, for $k \geq 2$,

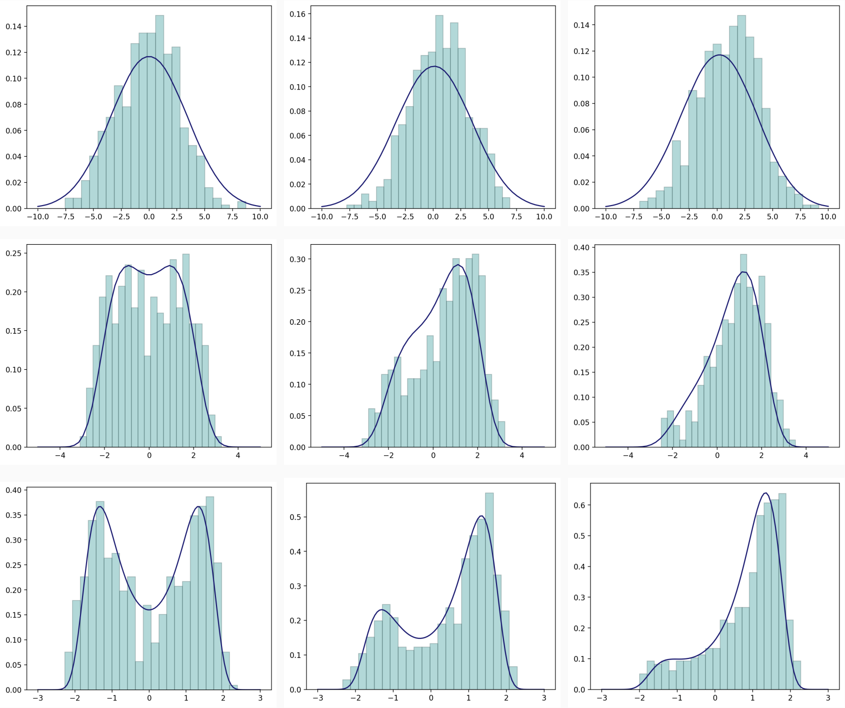
$$m_k = \frac{1}{ka - c_k} \sum_{j=1}^{k-1} c_j m_j m_{k-j},$$

where $c_k = 1$ for even k and $c_k = a$ for odd k. Then, for $k \geq 1$,

$$\mathbb{E}[L_1^k] = \frac{(k-1)!}{a\Gamma(ka)} m_k,$$

and the generating-moment function of L_1 is given by, for $t \in \mathbb{R}$,

$$\mathbb{E}[\mathrm{e}^{tL_1}] = \sum_{k\geq 0} \frac{m_k}{\Gamma(ka+1)} t^k.$$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Merci pour votre attention !