When is the convex hull of a Lévy path smooth?

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Figure: Path of a Brownian motion on $[0, 1]$ and the convex minorant C of its graph.

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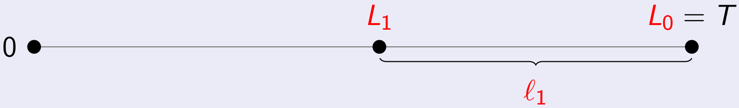

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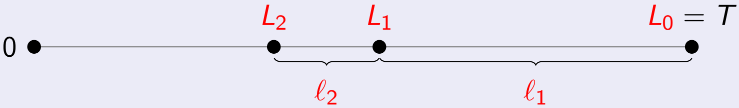

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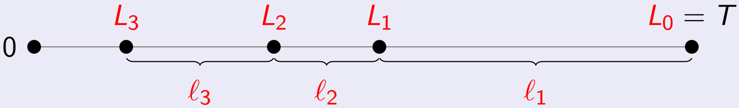

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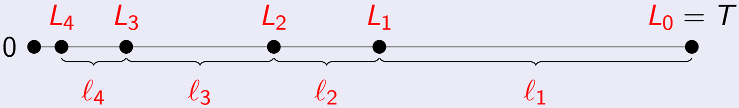

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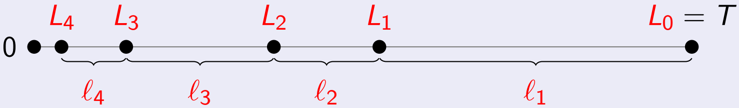

Then the faces of the convex minorant of X, sampled in a length-size-biased way (uniformly at random), have the same law as the sequence (ℓ_n, ξ_n), $n \in \mathbb{N}$, where, given ℓ, the variables $\xi_n = X_{L_{n-1}} - X_{L_n} \sim F(\ell_n, \cdot)$, $n \in \mathbb{N}$, are independent (here $F(t, \mathrm{d}x) = \mathbb{P}(X_t \in \mathrm{d}x)$ for $(t, x) \in (0, \infty) \times \mathbb{R}$).

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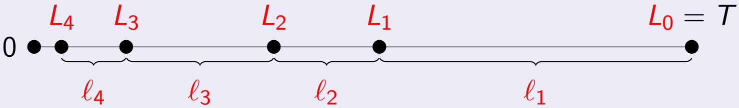

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Figure: Path of a Brownian motion on $[0, 1]$ and the convex minorant of its graph.

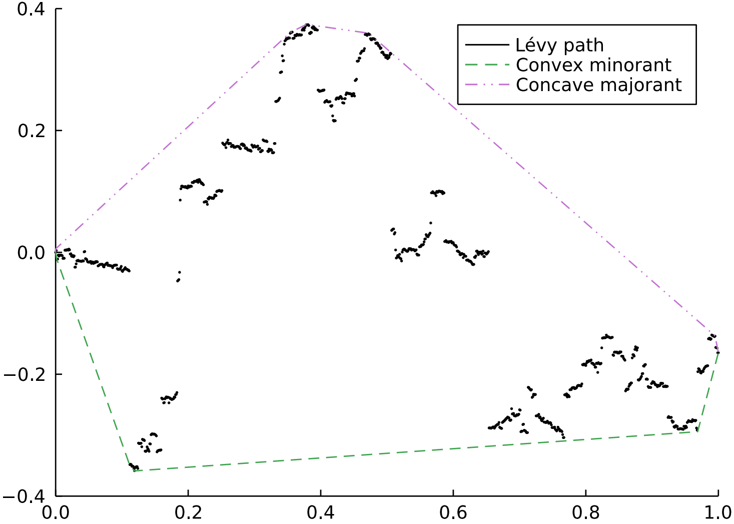
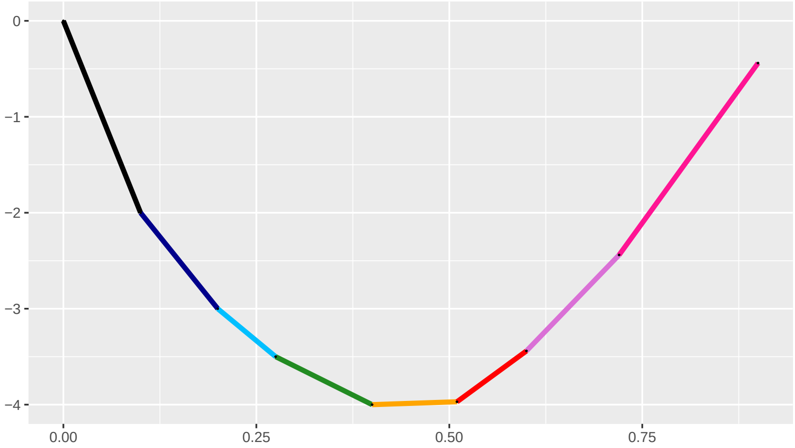


Figure: Path of a Lévy process on $[0,1]$ and the convex hull of its graph. (Cauchy case studied in [3])

Piecewise linear convex function and associated set of slopes



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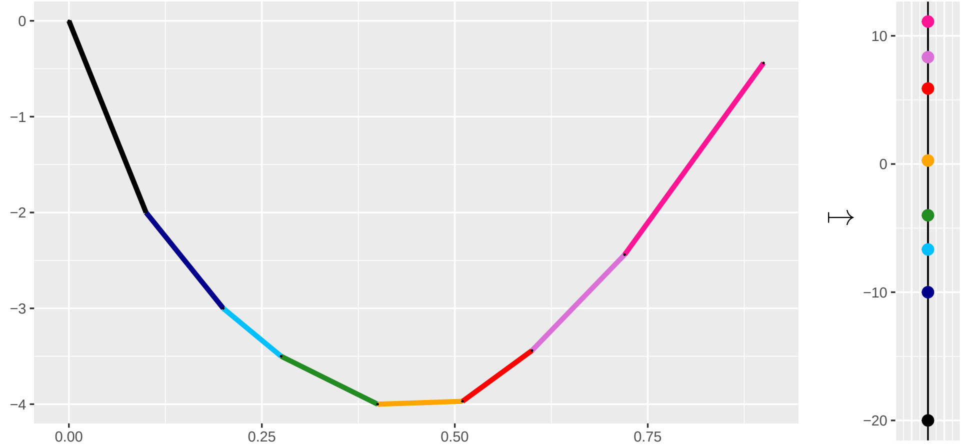

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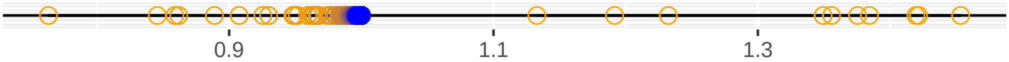
and corresponding set of slopes $\mathcal{S}$

Left, right and two-sided accumulation points of $\mathcal{S}$

Denote by $\mathcal{L}^-(\mathcal{S})$ (resp. $\mathcal{L}^+(\mathcal{S})$) the set of all left (resp. right) limit points of $\mathcal{S} \subset \mathbb{R}$.



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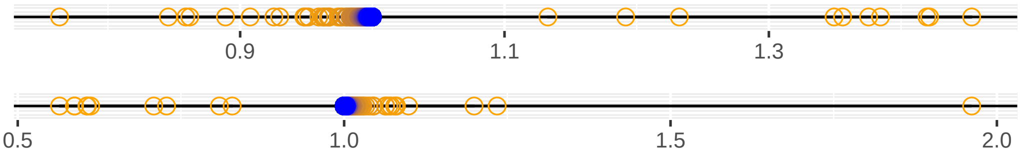

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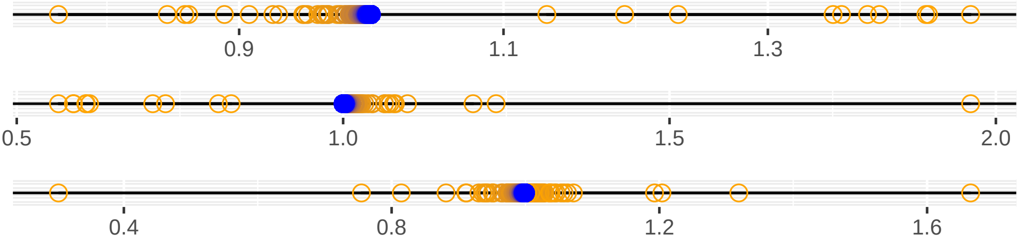

Let $\mathcal{L}(\mathcal{S}) := \mathcal{L}^-(\mathcal{S}) \cup \mathcal{L}^+(\mathcal{S})$ be the (closed) set of all limit points of $\mathcal{S}$.

Theorem 1 (B, González Cázares, Mijatović)

For any measurable set $I \subseteq \mathbb{R}$, the set $\mathcal{S} \cap I$ is either a.s. finite or a.s. infinite. Moreover, the cardinality $|\mathcal{S} \cap I|$ of the intersection $\mathcal{S} \cap I$ is infinite a.s. if and only if

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Theorem 2

The boundary of the convex hull of the graph $t \mapsto (t, X_t)$, $t \in [0, T]$, of a path of any Lévy process X is continuously differentiable (as a closed curve in $\mathbb{R}^2$) a.s. if and only if (1) holds for all intervals I in $\mathbb{R}$. Moreover, this is equivalent to the set $\mathcal{S}$ being dense in $\mathbb{R}$ a.s.

Finite variation X – results

Let $\psi(u) = \log \mathbb{E} e^{iuX_1} = iu\gamma_0 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(\mathrm{d}x)$ be the Lévy-Khintchine exponent of X.

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Lévy process X	Derivative C' and the limt set $\mathcal{L}(\mathcal{S})$
Finite variation (FV)	C' bounded below *and* above; C' discontinuous on boundary ∂I_r, $\forall r \in \mathcal{S}$; $\mathcal{L}(\mathcal{S}) = \{\gamma_0\}$, where $\gamma_0 = \lim_{t \downarrow 0} X_t/t$ a.s., and $\gamma_0 \notin \mathcal{S}$

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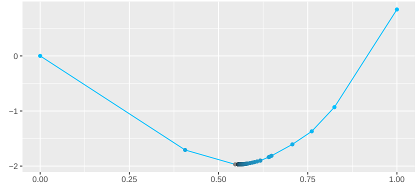
Behaviour depends on $I_- := \int_0^1 \mathbb{P}(X_t/t < \gamma_0)\frac{\mathrm{d}t}{t}$, $\quad I_+ := \int_0^1 \mathbb{P}(X_t/t > \gamma_0)\frac{\mathrm{d}t}{t}$ via

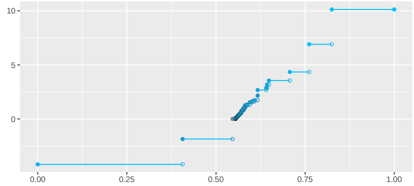
$$\mathcal{L}^{\pm}(\mathcal{S}) = \{\gamma_0\} \overset{\text{Thm 1}}{\iff} I_{\pm} = \infty \overset{[2]}{\iff} \int_{(-1,1)} \frac{\max\{\pm x, 0\}}{\int_0^{\max\{\pm x,0\}} \overline{\nu}_{\mp}(y)\mathrm{d}y}\nu(\mathrm{d}x) = \infty,$$

where $\overline{\nu}_+(x) := \nu((x,\infty))$ & $\overline{\nu}_-(x) := \nu((-\infty,-x))$, $x > 0$, and $\mp := -(\pm)$.

Convex minorant C (of an FV Lévy process) and its derivative C'

$I_- < \infty, \; I_+ = \infty$





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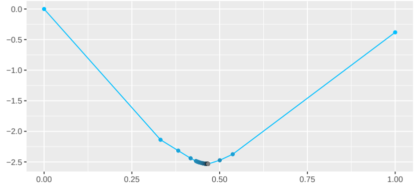
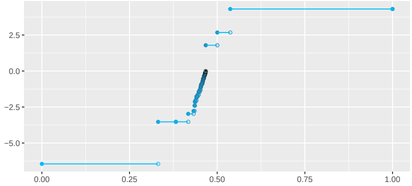
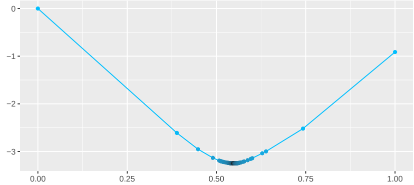
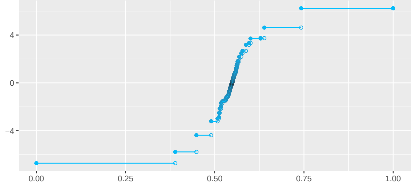




$I_+ = I_- = \infty$





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$\mathfrak{s}_1(r) = \infty$, $\forall r \in \mathbb{R}$	C' is continuous on $(0, T)$; $-\lim_{t\downarrow 0} C'(t) = \lim_{t\uparrow T} C'(t) = \infty$; $\mathcal{L}(\mathcal{S}) = \mathbb{R}$

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$r \in \mathcal{L}(\mathcal{S}) \iff \mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(r)$ *(i.e. $\mathfrak{s}_1$ not integrable on any neighbourhood of $r \in \mathbb{R}$).*

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Can these properties depend on r?

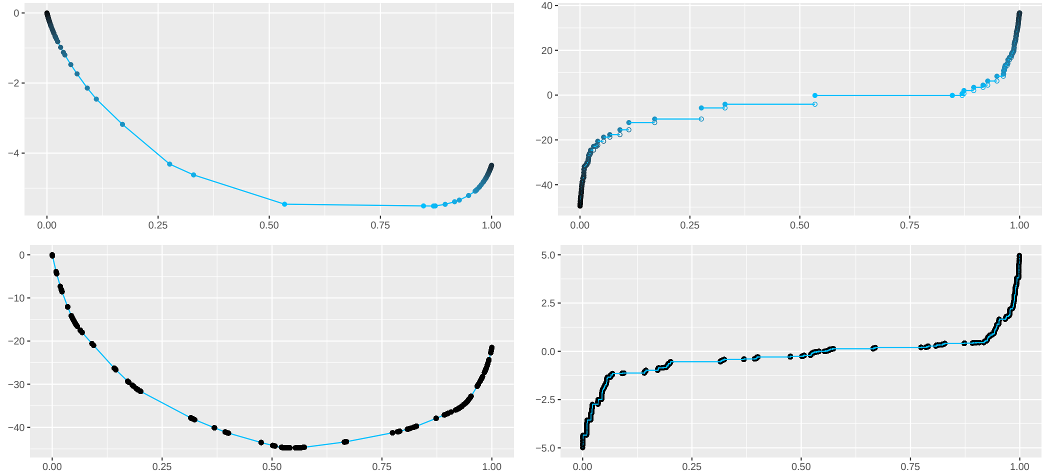
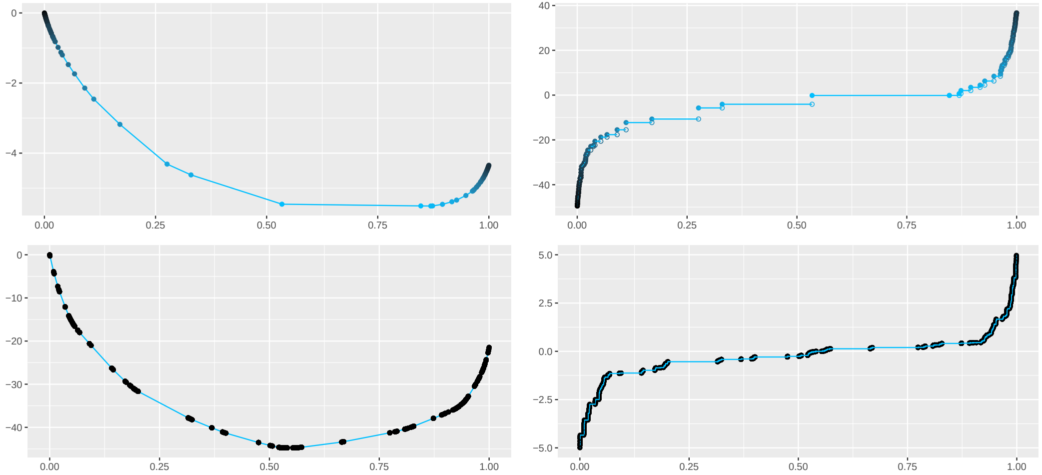


Figure: X of infinite variation (IV). Left: C; right: C'



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How can we tell which case it is in terms of the local behaviour of the paths of X?

Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s\uparrow t} X_s$ and define the left and right Dini derivatives:

$D_t^{\uparrow} := \limsup_{\varepsilon\uparrow 0}(X_{t+\varepsilon} - X_{t-})/\varepsilon$ and $D_t^{\downarrow} := \liminf_{\varepsilon\downarrow 0}(X_{t+\varepsilon} - X_t)/\varepsilon$.

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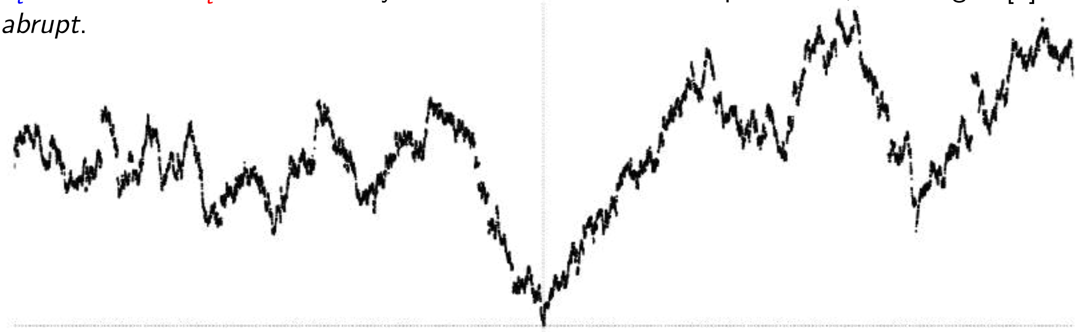
If $D_t^{\uparrow} = -\infty$ and $D_t^{\downarrow} = \infty$ at every local minimum t of an IV process X, then Vigon [6] calls X *abrupt*.



Proposition 2

A Lévy process X is abrupt if and only if $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s.

Infinite variation (IV)

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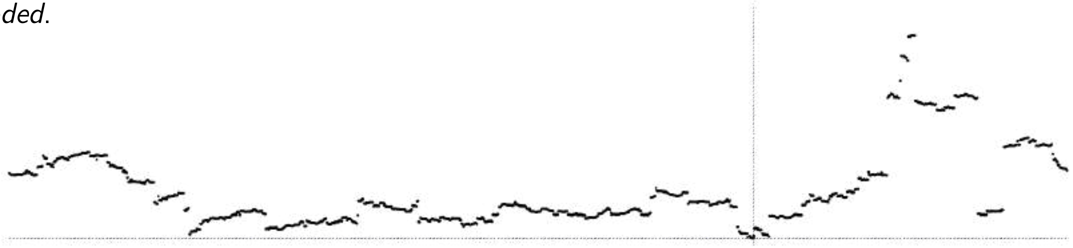
If $D_t^{\uparrow} = 0$ and $D_t^{\downarrow} = 0$ at every local minimum t of an IV process X, then Vigon [6] calls X *eroded*.



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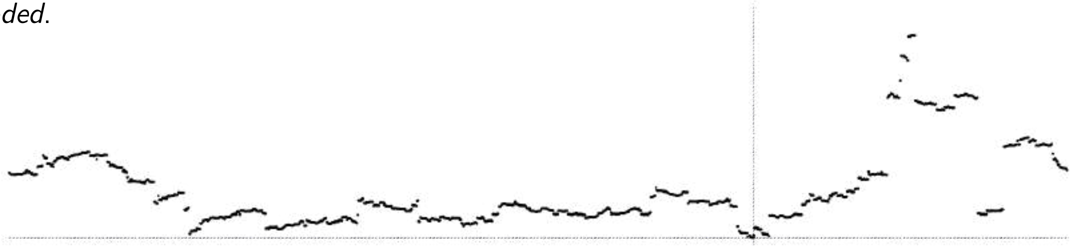

X is said to be *strongly eroded* if: $X^{(r)}=(X_t - rt)_{t\geq 0}$ is eroded $\forall r \in \mathbb{R}$.

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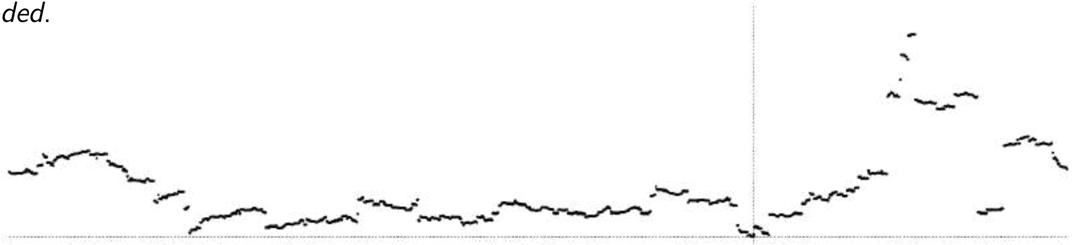
Abrupt (A) & Strongly Eroded (SE) Lévy processes

Denote $X_{t-} := \lim_{s \uparrow t} X_s$ and define the left and right Dini derivatives:
$D_t^{\uparrow} := \limsup_{\varepsilon \uparrow 0}(X_{t+\varepsilon} - X_{t-})/\varepsilon$ and $D_t^{\downarrow} := \liminf_{\varepsilon \downarrow 0}(X_{t+\varepsilon} - X_t)/\varepsilon$.

If $D_t^{\uparrow} = 0$ and $D_t^{\downarrow} = 0$ at every local minimum t of an IV process X, then Vigon [6] calls X *eroded*.



X is said to be *strongly eroded* if: $X^{(r)} = (X_t - rt)_{t \geq 0}$ is eroded $\forall r \in \mathbb{R}$.

Proposition 3

A Lévy process X is strongly eroded if and only if $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s.

Infinite variation (IV)

A conjectural dichotomy for IV Lévy processes

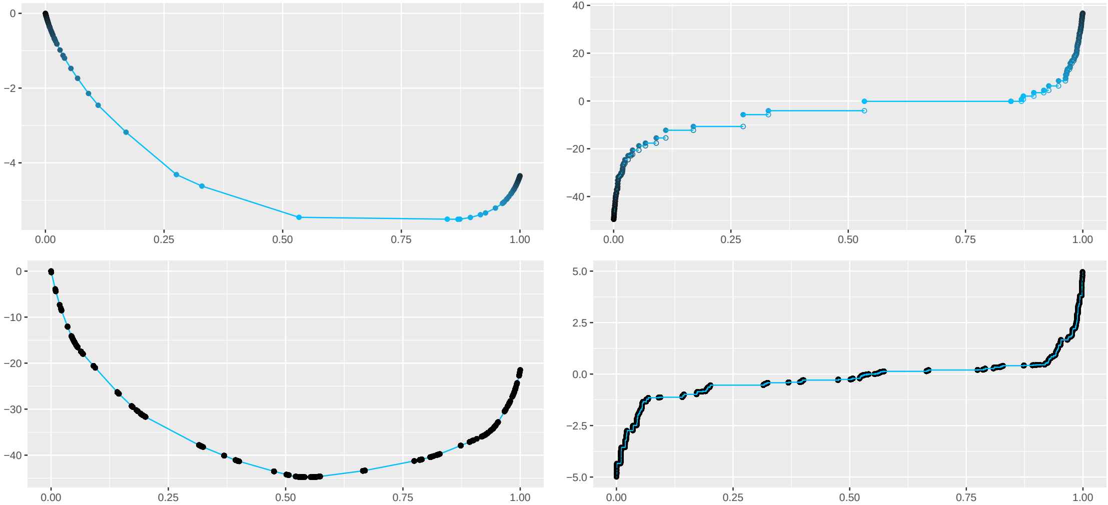


Figure: Convex minorant and its right derivative of an IV Lévy process X

A conjectural dichotomy

Conjecture 1

Any IV Lévy process is either A or SE. Equivalently, either $\mathfrak{s}_1 \in L^1_{\mathrm{loc}}(r)$, $\forall r \in \mathbb{R}$, *or* $\mathfrak{s}_1 = \infty$ *a.e.*

Geometrically, either the Lévy process shoots away from the convex minorant as soon as it touches it, or it stays close to the convex minorant when it touches it. Conjecture 1 is implied by Vigon's point-hitting conjecture:

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Conjecture 2 ([7, Conject. 1.6])

Let X *be an infinite variation process and for any* $r \in \mathbb{R}$ *define the Lévy process* $X^{(r)} = (X_t - rt)_{t \geq 0}$. *Then the following statements are equivalent.*

(i) *There exists some* $r \in \mathbb{R}$ *such that the process* $X^{(r)}$ *hits points.*

(ii) *For all* $r \in \mathbb{R}$ *the process* $X^{(r)}$ *hits points.*

(iii) *The process* X *is abrupt.*

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(i) *There exists some* $r \in \mathbb{R}$ *such that the process* $X^{(r)}$ *hits points.* $(\iff \mathfrak{s}_1(r) < \infty.)$

(ii) *For all* $r \in \mathbb{R}$ *the process* $X^{(r)}$ *hits points.* $(\iff \mathfrak{s}_1(r) < \infty, \forall r \in \mathbb{R}.)$

(iii) *The process* X *is abrupt.* $(\iff \mathfrak{s}_1 \in L^1_{\mathrm{loc}}(r), \forall r \in \mathbb{R}.)$

Infinite variation X – results

The set of slopes $\mathcal{S}$ is unbounded on both sides for any X of IV, i.e. $\sup \mathcal{S} = -\inf \mathcal{S} = \infty$, and hence $-\lim_{t\downarrow 0} C'(t) = \lim_{t\uparrow T} C'(t) = \infty$ a.s.

Any SE (resp. A) process, when perturbed by a finite variation process, is still SE (resp. A).

Proposition 4

Suppose $X = Y + Z$ for (possibly dependent) Lévy processes Y and Z. Let $\mathcal{S}_X$ and $\mathcal{S}_Z$ be the sets of slopes of the faces of the convex minorants of X and Z, respectively. If Y is of FV (possibly finite activity) with natural drift b, then $\mathcal{L}(\mathcal{S}_X) = \mathcal{L}(\mathcal{S}_Z) + b$.

Recipe to construct many SE and A processes!

Let Z be standard Cauchy process, which is SE since law of Z_t/t does not depend on t (first proved by Bertoin [3])

Too much asymmetry breaks smoothness

Proposition 5

If X is IV with $\nu((-y,-x]) \geq c\nu([x,y))$ for some $c > 1$ and and all $0 < x < y$ close to zero, then $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. making X abrupt.

Example:

Weakly 1-stable process, i.e. $\nu((-\infty,-x)) = c_-x^{-1}$ and $\nu((x,\infty)) = c_+x^{-1}$ for all $x > 0$ and some $c_+ \neq c_-$.

Sufficient conditions for X to be strongly eroded (or abrupt)

Corollary 2

Let X be a Lévy process of IV with $e^{\psi(u)} = \mathbb{E}e^{iuX_1}$, $u \in \mathbb{R}$.

(i) *If $\limsup_{u\to\infty} |\psi(u)/u| < \infty$, then X is SE.*

(ii) *If $\lim_{u\to\infty} |\psi(u)/u| = \infty$, then X is either A or SE.*

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In fact, if $\liminf_{u\to\infty} |\psi(u)/u^{1+\varepsilon}| > 0$ for some $\varepsilon > 0$, then X is A.

Most Lévy processes are included! Examples:

(a) Any process attracted to an α-stable process in small-time with $\alpha \in (1, 2]$;

(b) 1-semi-stable processes is SE if it is strictly 1-semi-stable and otherwise it is A.

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It excludes Lévy processes with $\liminf_{|u|\to\infty} |\psi(u)/u| < \infty = \limsup_{|u|\to\infty} |\psi(u)/u|$, e.g., Orey's process (singular continuous IV process with purely atomic Lévy measure with Blumenthal–Getoor index $\beta_+ \in (1, 2)$ and $\mathfrak{s}_1(0) = \infty$).

Infinite variation (IV)

Domain of attraction to Cauchy process

It is known that the convex hull of a Cauchy process is SE [3]. Similarly, processes in the domain of *normal* attraction are also SE:

Example 7.1

* If $X_t/t \xrightarrow{d} S$ as $t \downarrow 0$ for some Cauchy random variable S (so-called *normal* attraction, e.g. $\nu([x,\infty))x \to c$, $\nu((-\infty,-x])x \to c$ and $\int_{(-1,-x]\cup[x,1)} y\nu(\mathrm{d}y) \to c'$ as $x \downarrow 0$ for some $c > 0$ and $c' \in \mathbb{R}$), then X is SE.
* If $X_t/(tg(t)) \xrightarrow{d} S$ as $t \downarrow 0$ for a slowly varying g at 0, then C may be either SE or A. Depends on the size of the fluctuations of g!

Sums of independent abrupt (A) and strongly eroded (SE) processes

Any of the following are possible for independent summands are realises:

* A + A = A,
* A + A = SE,
* A + SE = A,
* A + SE = SE,
* SE + SE = SE,
* SE + SE = A.

See paper :
D. Bang, J. G. C., A. Mijatović. "When is the convex hull of a Lévy path smooth?" (2022). To appear in Annales de l'Institute Henri Poincaré. `https://arxiv.org/abs/2205.1441`

Growth of the derivative C'

Regimes:	Finite slope (FS)	
Setting:	$s \in \mathcal{L}^+(\mathcal{S})$ a.s., i.e. C' a.s. non-constant at vertex time τ_s	
Upper functions:	$\limsup_{t \downarrow 0}(C'_{t+\tau_s} - s)/f(t)$	
Lower functions:	$\liminf_{t \downarrow 0}(C'_{t+\tau_s} - s)/f(t)$	

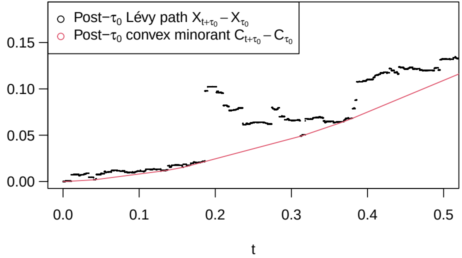
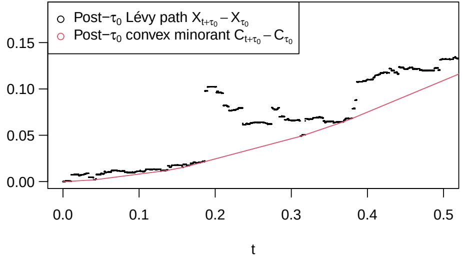


Figure: (FS) regime.

Growth of the derivative C'

Regimes:	Finite slope (FS)	Infinite slope (IS)
Setting:	$s \in \mathcal{L}^+(\mathcal{S})$ a.s., i.e. C' a.s. non-constant at vertex time τ_s	IV X, i.e. $\lim_{t\downarrow 0} C'_t = -\infty$ and non-constant C' at time 0
Upper functions:	$\limsup_{t\downarrow 0}(C'_{t+\tau_s} - s)/f(t)$	$\limsup_{t\downarrow 0} \lvert C'_t \rvert f(t)$
Lower functions:	$\liminf_{t\downarrow 0}(C'_{t+\tau_s} - s)/f(t)$	$\liminf_{t\downarrow 0} \lvert C'_t \rvert f(t)$



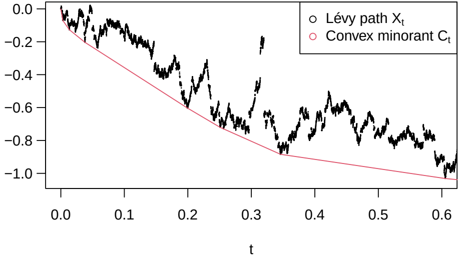


Figure: Left: (FS) regime. Right: (IS) regime.

Regime (FS): Behaviour at vertex time τ_s

Corollary 8.1

Suppose $x^\alpha \nu([x,\infty)) \to c_+ > 0$ *and* $x^\alpha \nu((-\infty,-x]) \to c_-$ *for some* $\alpha \in (0,1)$*, and denote* $\rho := \lim_{t\downarrow 0} \mathbb{P}(X_t > \gamma_0 t) \in (0,1]$ *where* $\gamma_0 := \lim_{t\downarrow 0} X_t/t$*. Set* $f(t) := t^{1/\alpha-1}\log^q(1/t)$*, then:*

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(iii) $\limsup_{t\downarrow 0}(C'_{t+\tau_s} - s)/f(t) = 0$ *for* $q > (1/\alpha - 1)/\rho$.

D. Bang, J. G. González Cázares, A. Mijatović. " How smooth can the convex hull of a Lévy path be" (2022). https://arxiv.org/abs/2206.09928

Is C r-Hölder continuous ($\sup_{0\le u<t\le T}\frac{|C_t-C_u|}{(t-u)^r}<\infty$ a.s.)?

Blumenthal–Getoor index of X is critical: $\beta := \inf\left\{p>0 \,:\, \int_{(-1,1)}|x|^p\nu(\mathrm{d}x)<\infty\right\}$

<table>
<tr><th colspan="3">Lévy process X</th><th>$r\in(0,1]$</th><th>Is C r-Hölder continuous?</th></tr>
<tr><td colspan="3" rowspan="2">$\sigma^2>0$</td><td>$0<r<1/2$</td><td>Yes</td></tr>
<tr><td>$1/2\le r\le 1$</td><td>No</td></tr>
<tr><td rowspan="8">$\sigma^2=0$</td><td colspan="2">$\beta\in[0,1]$ and FV</td><td>$0<r\le 1$</td><td>Yes</td></tr>
<tr><td colspan="2" rowspan="2">$\beta=1$ and IV</td><td>$0<r<1$</td><td>Yes</td></tr>
<tr><td>$r=1$</td><td>No</td></tr>
<tr><td rowspan="4">$\beta\in(1,2]$</td><td rowspan="2">$\int_{(-1,1)}|x|^\beta\nu(\mathrm{d}x)=\infty$</td><td>$0<r<1/\beta$</td><td>Yes</td></tr>
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<tr><td rowspan="2">$I_\beta<\infty$</td><td>$0<r\le 1/\beta$</td><td>Yes</td></tr>
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</table>

where $I_\beta := \int_0^1\mathbb{E}\left[\min\{|X_t|/t^{1/\beta},1\}^{\beta/(\beta-1)}\right]\frac{\mathrm{d}t}{t}$ for $\beta\in(1,2]$.

D. Bang, J. G. González Cázares, A. Mijatović. "Hölder continuity of the convex minorant of a Lévy process" (2022). https://arxiv.org/abs/2207.12433

[1] David Bang, Jorge González Cázares, and Aleksandar Mijatović. "When is the convex hull of a Lévy path smooth?" (2022). DOI: 10.48550/ARXIV.2205.14416.

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