Quantitative mean-field limit for interacting branching diffusions

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Outline

1. Motivation: Lotka-Volterra cross diffusion models

2. Mean-field interacting branching diffusions

3. Main result: convergence rate

4. Ideas of the proof: coupling and optimal transport

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> Global (weak) solutions: 📄 [Galiano, Garzón, Jüngel 2003],
📄 [Chen, Jüngel 2004-2006], 📄 [Chen, Daus, Jüngel 2018], 📄 [Chen, Jüngel, Wang 2022]. Entropic structure.

> Non-local SKT system introduced in 📄 [F', Méléard 2016] :

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\partial_t u - \Delta(d_1 u + a_{11}(u * G^{1,1})u + a_{12}(v * G^{1,2})u) = (r_1 - s_{11}u * C^{1,1} - s_{12}v * C^{1,2})u, \\
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> #### Question 🔍
> Can we quantify the approximation of non-local SKT system by individual based models?

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> Between birth and death events the individual $X^{n,K}$ evolves as the diffusion process
> $$\mathrm{d}X_t^{n,K} = b\big(X_t^{n,K}, H * \mu_t^K(X_t^{n,K})\big)\,\mathrm{d}t + \sigma\big(X_t^{n,K}, G * \mu_t^K(X_t^{n,K})\big)\,\mathrm{d}B_t^n.$$

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Theorem (F', Méléard '16, particular case)

Under Lipschitz regularity on σ, b, G, and H, moment assumptions on $(\mu_0^K)_K$ and weak convergence $\mu_0^K \to \mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$, processes $(\mu_t^K)_{t\geq 0}$ converge weakly as $K \to \infty$ to $(\mu_t)_{t\geq 0}$, the unique weak solution of

$$\partial_t \mu_t = L_{\mu_t}^* \mu_t + (r - c\langle \mu_t, 1\rangle)\mu_t,$$

with i.c. μ_0, where

$$L_\mu f(x) = \frac{1}{2}\mathrm{Tr}\left(a(x, G * \mu(x))\mathrm{Hess}(f)(x)\right) + b(x, H * \mu(x)) \cdot \nabla f(x).$$

Remarks:

> $(N_t^K/K)_{t\geq 0}$ converges to $n_t := \langle \mu_t, 1\rangle$ solution of **logistic** ODE $\dot{n}_t = n_t(r - cn_t)$

> $\bar{\mu}_t := \mu_t/\langle \mu_t, 1\rangle$ satisfies **nonlinear diffusion equation of McKean-Vlasov type**:

$$\partial_t \bar{\mu}_t = L_{\mu_t}^* \bar{\mu}_t$$

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> ❯ Conditionally on N_0^K, atoms of μ_0^K are i.i.d. $\sim \bar{\mu}_0 := \mu_0/\langle\mu_0, 1\rangle$, $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$

> ❯ $\int_{\mathbb{R}^d} |x|^q \, \mu_0(\mathrm{d}x) < \infty$ for some $q > 2$ and $\sup_K \mathbb{E}(\langle\mu_0^K, 1\rangle^4) < \infty$.

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Theorem (F., Muñoz, 2022, EJP)

For all $K, T > 0$ one has

$$\sup_{t \in [0,T]} \mathbb{E}\big(\big\|\mu_t^K - \mu_t\big\|_{\mathrm{BL}^*}\big) \leq C_T \left(I_4(K) + R_{q,d}(K) \right)$$

where $C_T > 0$, $R_{q,d}(K)$ is an explicit polynomial function $\to 0$ as $K \to \infty$ and

$I_4(K) = \mathbb{E}\big(|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle|^4\big)^{\frac{1}{4}}$ *is a smaller term.*

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More important than making $R_{q,d}(K)$ explicit is.....

Where does the rate $R_{d,q}(K)$ come from?

Recall:

> **Wasserstein distance:** $p \in [1, \infty)$, p-Wasserstein distance $W_p(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ defined by

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}}.$$

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For any $X \sim \mu$, $Y \sim \nu$ on common probability space, $W_p(\mu, \nu) \leq \mathbb{E}(|X - Y|^p)^{\frac{1}{p}}$

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› 📄 [Fournier, Guillin 2015]: **Quantitative L.L.N. for empirical distributions :**
$(Z_i)_{i \geq 1}$ i.i.d. $\sim m \in \mathcal{P}(\mathbb{R}^d)$ with $q-$moment $M_q < \infty$, $q > 2$.

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> 📄 [Fournier, Guillin 2015]: **Quantitative L.L.N. for empirical distributions :**
> $(Z_i)_{i \geq 1}$ i.i.d. $\sim m \in \mathcal{P}(\mathbb{R}^d)$ with $q-$moment $M_q < \infty$, $q > 2$.
> Then, for some $C_{d,q} > 0$ and all $N \in \mathbb{N} \setminus \{0\}$,
>
> $$\mathbb{E}\left(W_2^2 \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i}, m \right) \right) \leq C_{d,q} M_q^{\frac{2}{q}}(m) \, R_{d,q}^2(N).$$

For all $K > 0$ one has

$$R_{d,q}(K) = \begin{cases} K^{-\frac{1}{4}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d < 4 \text{ and } q \neq 4, \\ K^{-\frac{1}{4}}(\log(1+K))^{\frac{1}{2}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d = 4 \text{ and } q \neq 4, \\ K^{-\frac{1}{d}} + K^{-\frac{(q-2)}{2q}}, & \text{if } d > 4 \text{ and } q \neq \frac{d}{(d-2)}, \end{cases}$$

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❯ Bound easily obtained at $t = 0$ from Fournier-Guillin's result and conditional independence of atoms of μ_0^K given mass $\frac{N_0^K}{K}$:

$$\mathbb{E}\left(\|\mu_0^K - \mu_0\|_{\mathrm{BL}^*}\right) \leq C \, \mathbb{E}\left(\frac{N_0^K}{K} W_2^2\left(\bar{\mu}_0^K, \bar{\mu}_0\right)\right)^{\frac{1}{2}} \mathbb{E}\left(\frac{N_0^K}{K}\right)^{\frac{1}{2}} + \mathbb{E}\left(|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle|\right),$$
$$\leq C \, R_{q,d}(K) + \mathbb{E}\left(|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle|\right)$$

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> If $\bar{\mu}, \bar{\nu} \in \mathcal{P}(\mathbb{R}^d)$ are normalized versions of $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$, then

$$\|\mu - \nu\|_{\mathrm{BL}^*} \le \langle \mu, 1 \rangle W_1(\bar{\mu}, \bar{\nu}) + |\langle \mu, 1 \rangle - \langle \nu, 1 \rangle|$$

> Bound easily obtained at $t = 0$ from Fournier-Guillin's result and conditional independence of atoms of μ_0^K given mass $\frac{N_0^K}{K}$:

$$\mathbb{E}\left(\|\mu_0^K - \mu_0\|_{\mathrm{BL}^*}\right) \le C \, \mathbb{E}\left(\frac{N_0^K}{K} W_2^2(\bar{\mu}_0^K, \bar{\mu}_0)\right)^{\frac{1}{2}} \mathbb{E}\left(\frac{N_0^K}{K}\right)^{\frac{1}{2}} + \mathbb{E}\left(|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle|\right),$$

$$\le C \, R_{q,d}(K) + \mathbb{E}\left(|\langle \mu_0^K, 1 \rangle - \langle \mu_0, 1 \rangle|\right)$$

No longer true for $t > 0$!!!

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$$\frac{N_t^K}{K} W_2^2(\bar{\mu}_t^K, \bar{\mu}_t) \leq 2\frac{N_t^K}{K} W_2^2(\bar{\nu}_t^K, \bar{\mu}_t) + 2\frac{N_t^K}{K} W_2^2(\bar{\mu}_t^K, \bar{\nu}_t^K).$$

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$$\mathrm{d}Y_t^{n,K} = b\big(Y_t^{n,K}, H * \mu_t(Y_t^{n,K})\big) \, \mathrm{d}t + \sigma\big(Y_t^{n,K}, G * \mu_t(Y_t^{n,K})\big) \, \mathrm{d}B_t^n.$$

so that $\mathsf{Law}(Y_t^{n,K}) = \bar{\mu}_t$, and we must chose $Y_\tau^{n,K} \sim \bar{\mu}_\tau$ at its birth time τ.

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> Term $\frac{N_t^K}{K} W_2^2(\bar{\nu}_t^K, \bar{\mu}_t)$: OK similarly as for $t = 0$.

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
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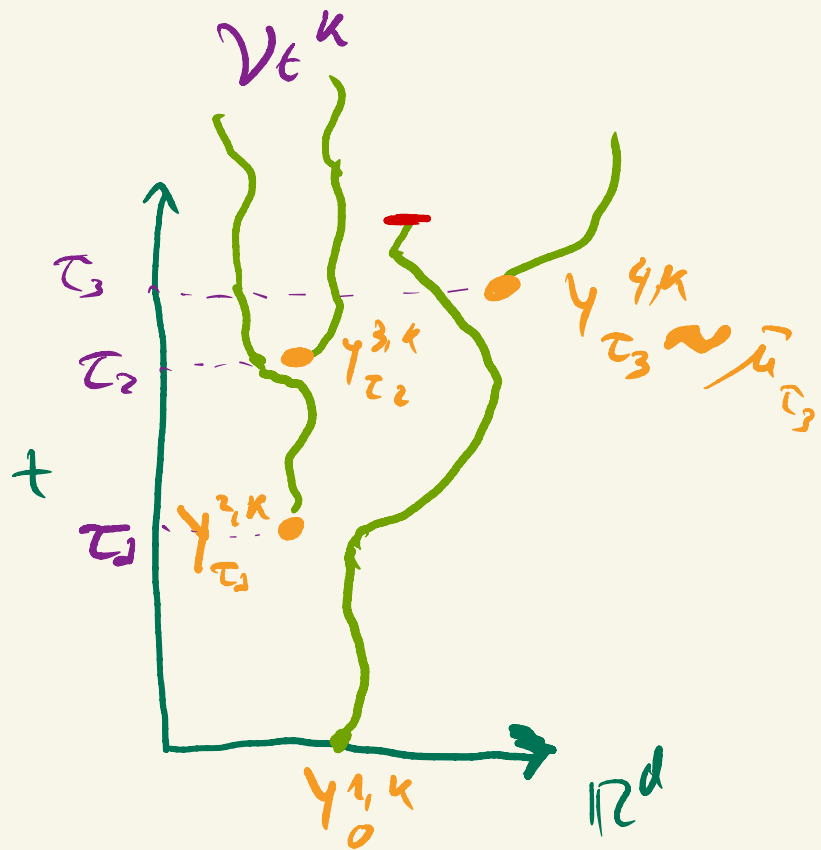- Term $\frac{N_t^K}{K} W_2^2(\bar{\nu}_t^K, \bar{\mu}_t)$: OK similarly as for $t = 0$.

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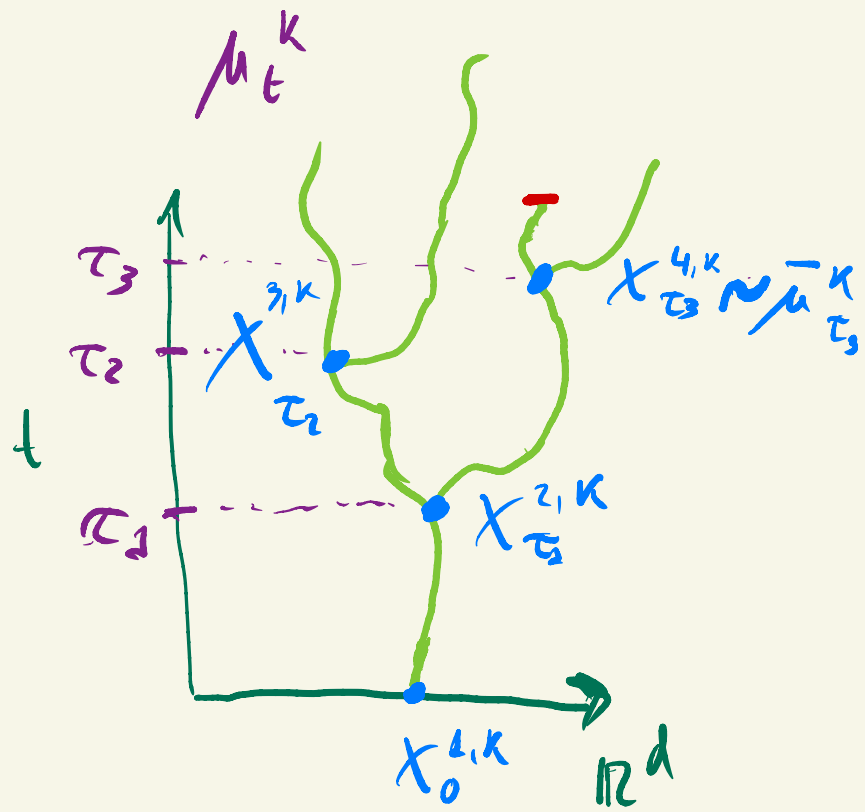
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> Drawing

> Triangle ineq. with $\bar{\nu}_{\tau-}^K$ + Gronwall $\Rightarrow$ remainder terms of order $R_{d,q}(K)$ too.

μ_t^K

τ_3

τ_2

t

τ_1

$X^{3,K}_{\tau_2}$

$X^{4,K}_{\tau_3} \sim \bar{\mu}^K_{\tau_3}$

$X^{2,K}_{\tau_1}$

$X^{1,K}_0$

$\mathbb{R}^d$

ν_t^K

τ_3

τ_2

t

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Thank you!