The standard augmented multiplicative coalescent revisited

Josué Corujo Rodríguez

work in collaboration with Vlada Limic

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Institut de Recherche Mathématique Avancée

A multi-graph valued Markov process

$\big(MG^{(n)}(t), t \geq 0\big)$ multi-graph-valued continuous-time Markov chain

- n vertices and i-th vertex has size x_i
- number of edges $i \to j$ at time $t : N_{\{i,j\}}(t)$ → Poisson process with rate $x_i \cdot x_j$

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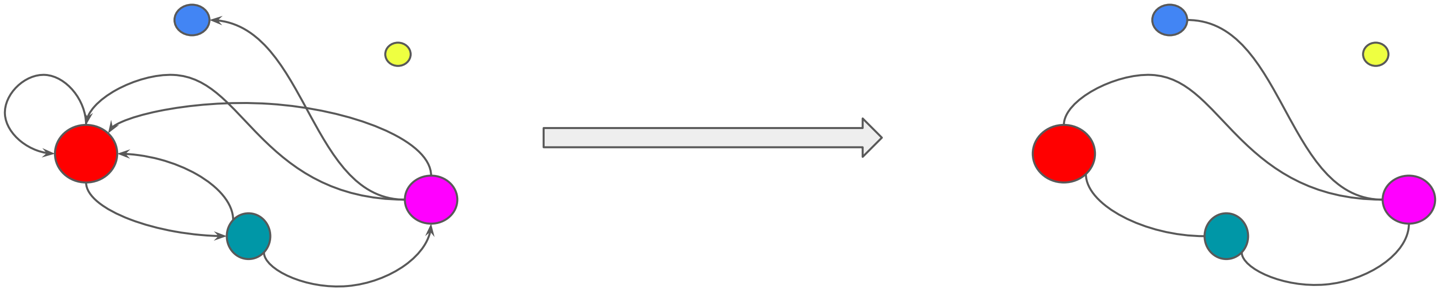
- n vertices and i-th vertex has size x_i
- number of edges $i \to j$ at time $t : N_{\{i,j\}}(t)$

Poisson process with rate $x_i \cdot x_j$

removing multi-edges and self-loops

Inhomogeneous random graph:

$$\mathbb{P}[\{i,j\} \in G^{(n)}(t)] = 1 - \mathrm{e}^{-t \cdot x_i \cdot x_j}$$



The multiplicative coalescent (MC)

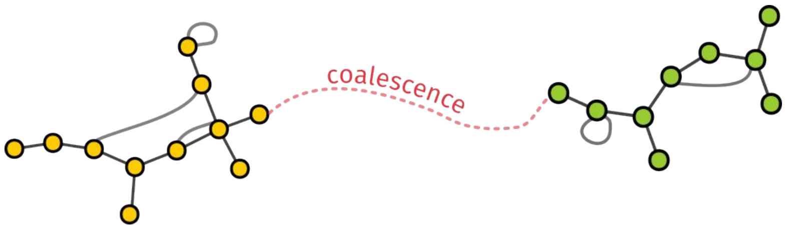
$\left(MG^{(n)}(t), t \geq 0\right)$ multi-graph-valued continuous-time Markov chain

$C_i^{(n)}(t)$ i-th largest component of $MG^{(n)}(t)$ and also $G^{(n)}(t)$

$\Longrightarrow$ $\left((|C_1^{(n)}(t)|, |C_2^{(n)}(t)|, \dots), t \geq 0\right)$ Markov process with MC dynamic

$$X_i, \dots, X_j \rightsquigarrow X_i + X_j$$

with rate $X_i \cdot X_j$



The MC dynamic is encoded by the excursions of a Lévy-type process

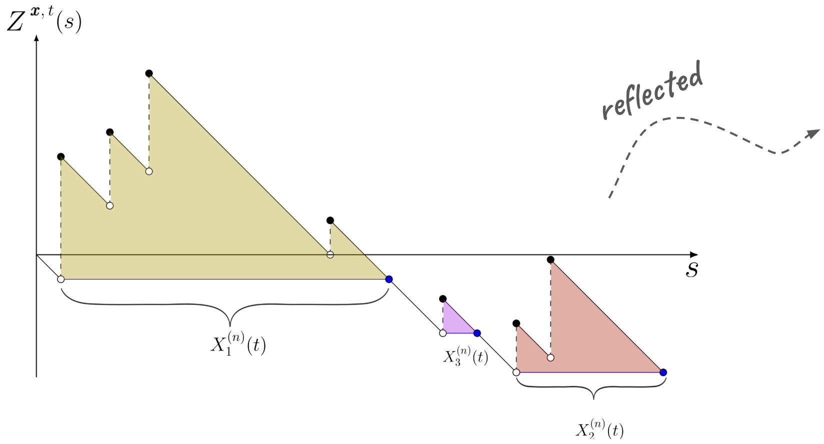
The simultaneous breadth-first walk

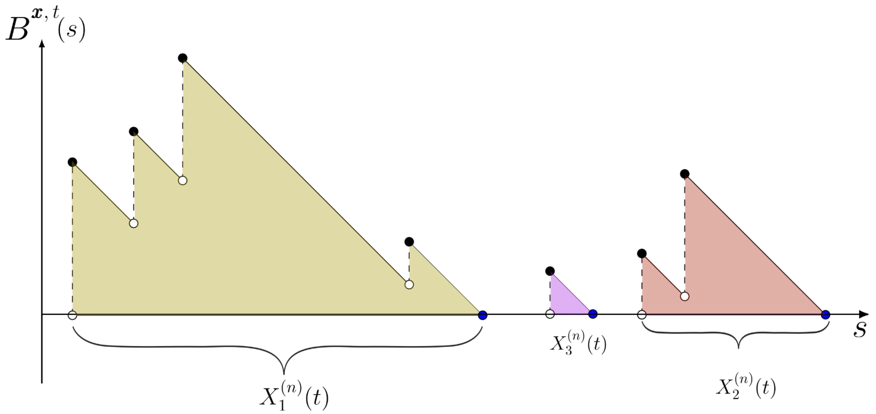
The multiplicative coalescent is encoded by the excursions above infima of

$$Z^{\boldsymbol{x},t}(s) := \sum_{i=1}^{n} x_i \cdot \mathbf{1}_{\{\xi_i/t \le s\}} - s \qquad \text{where} \qquad \xi_i \sim \mathrm{Exp}(\text{rate } = x_i)$$

or equivalently by the excursions above zero of the reflected process

$$B^{\boldsymbol{x},t}(s) := Z^{\boldsymbol{x},t}(s) - \inf_{0\le u\le s} Z^{\boldsymbol{x},t}(u)$$





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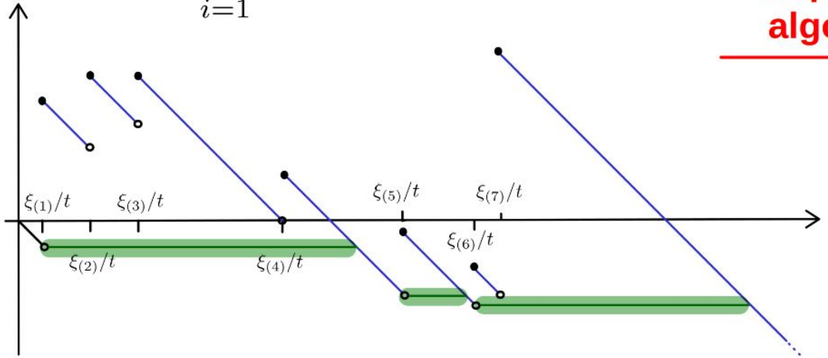
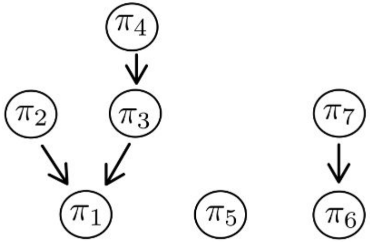


breadth-first exploration algorithm

random forest



component (trees) sizes

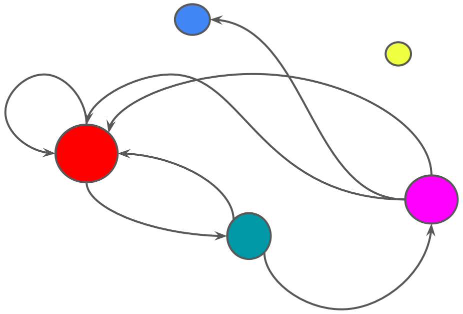
$$X(t) = (x_{\pi_1} + \cdots + x_{\pi_4}, x_{\pi_6} + x_{\pi_7}, x_{\pi_5})$$

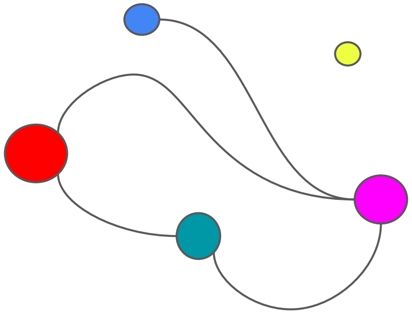
Limic (2019)

$(\boldsymbol{X}(t), t \ge 0)$ is a multiplicative coalescent started from $\boldsymbol{x}$

Our results

⟹ Enrich the encoding to account for the sizes of the connected components and the **number of surplus edges** (augmented multiplicative coalescent)





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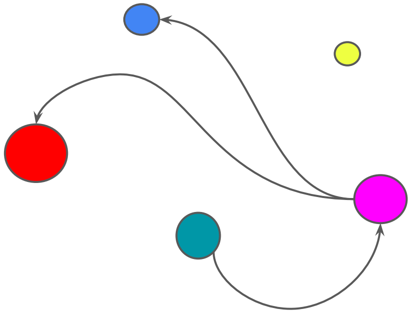

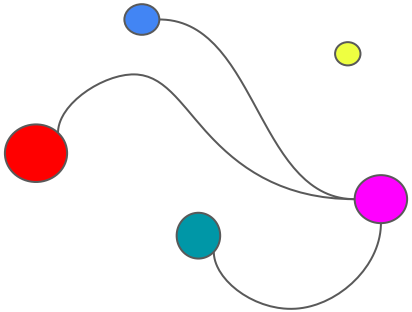

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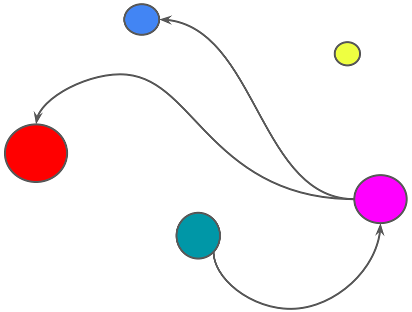

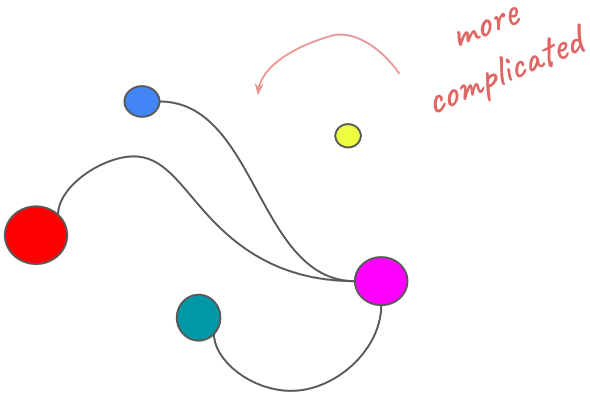

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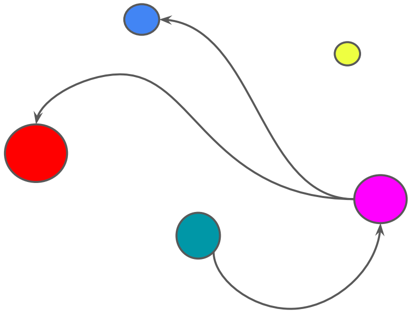

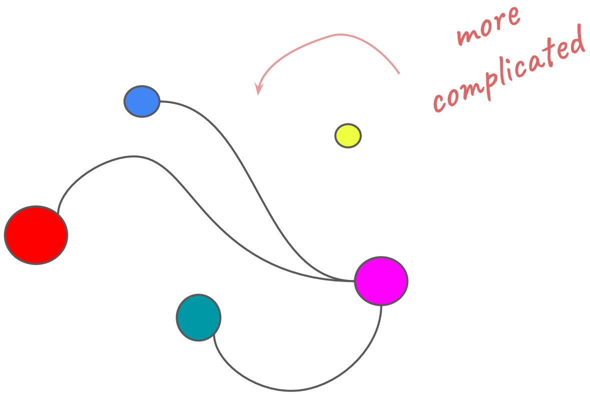

⟹ Study the scaling limits of the augmented multiplicative coalescent

Our results

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arXiv > math > arXiv:2305.04716

Mathematics > Probability

[Submitted on 8 May 2023]

A dynamical approach to spanning and surplus edges of random graphs

Josué Corujo, Vlada Limic

⟹ Study the scaling limits of the augmented multiplicative coalescent

arXiv > math > arXiv:2304.07545

Mathematics > Probability

[Submitted on 15 Apr 2023 (v1), last revised 9 May 2023 (this version, v2)]

The standard augmented multiplicative coalescent revisited

Josué Corujo, Vlada Limic

The augmented multiplicative coalescent (AMC)

$\left(MG^{(n)}(t), t \geq 0\right)$ multi-graph-valued continuous-time Markov chain

$C_i^{(n)}(t)$ size of the i-th largest component of $MG^{(n)}(t)$ and also $G^{(n)}(t)$

$\mathrm{SP}(C_i^{(n)}(t))$ number of surplus edges in the i-th largest component of $MG^{(n)}(t)$

$\Longrightarrow \left(\left(|C_i^{(n)}(t)|, \mathrm{SP}\left(C_i^{(n)}(t)\right)\right)_{i\geq 1}, t \geq 0\right)$ Markov process with AMC dynamic:

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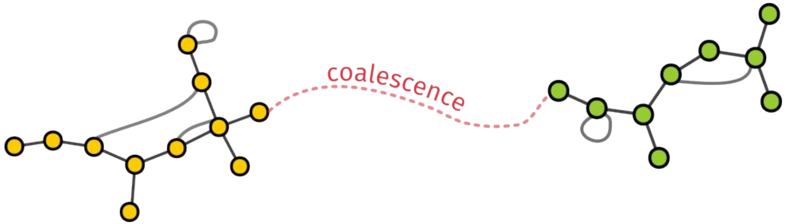
$$(X_i, N_i), \ldots, (X_j, N_j) \rightsquigarrow (X_i + X_j, N_i + N_j)$$

with rate $X_i \cdot X_j$



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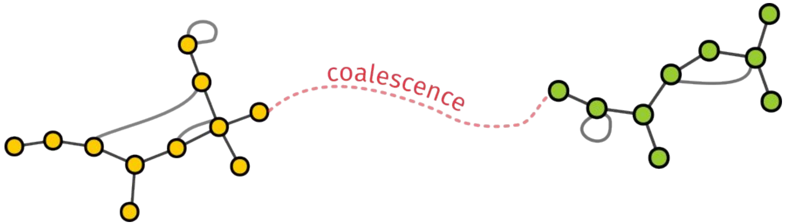
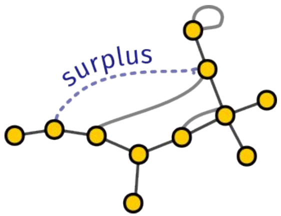


$$(X_i, N_i) \rightsquigarrow (X_i, N_i + 1)$$

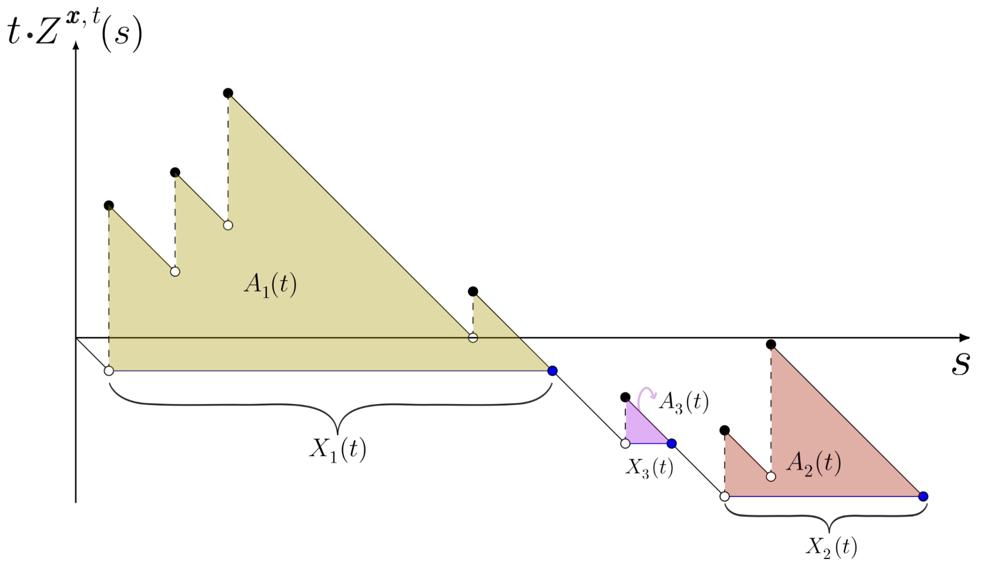
with rate $X_i^2/2$



The encoding of the AMC

$$Z^{\boldsymbol{x},t}(s) := \sum_{i=1}^{n} x_i \cdot \mathbf{1}_{\{\xi_i/t \le s\}} - s \quad \text{where} \quad \xi_i \sim \mathrm{Exp}(\text{rate} = x_i)$$

$$N_i(t) = \mathrm{Poisson}(A_i(t))$$



C. and Limic (2023+)

$$\big(X_i(t), N_i(t)\big)_{i \ge 1}$$

has the law of an augmented multiplicative coalescent started from $(\boldsymbol{x}, \mathbf{0})$, at time t

An application

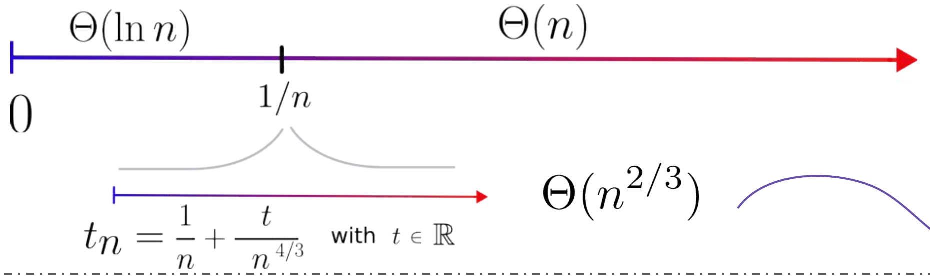
Erdös-Rényi random graph:

$$\mathbb{P}[\{i,j\} \in G(n,p)] = p$$

independently, for every pair of vertices

$C_i^{(n)}$ i-th largest component of $G(n,p)$

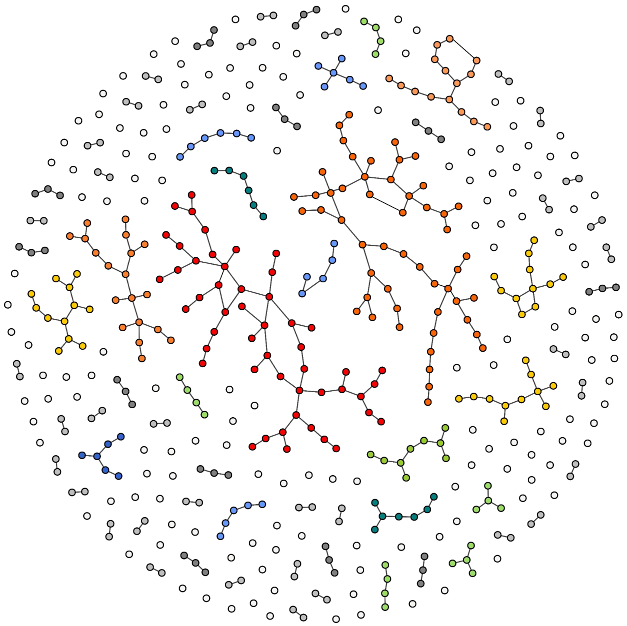
Erdös-Rényi (1960), Bollobás (1985)



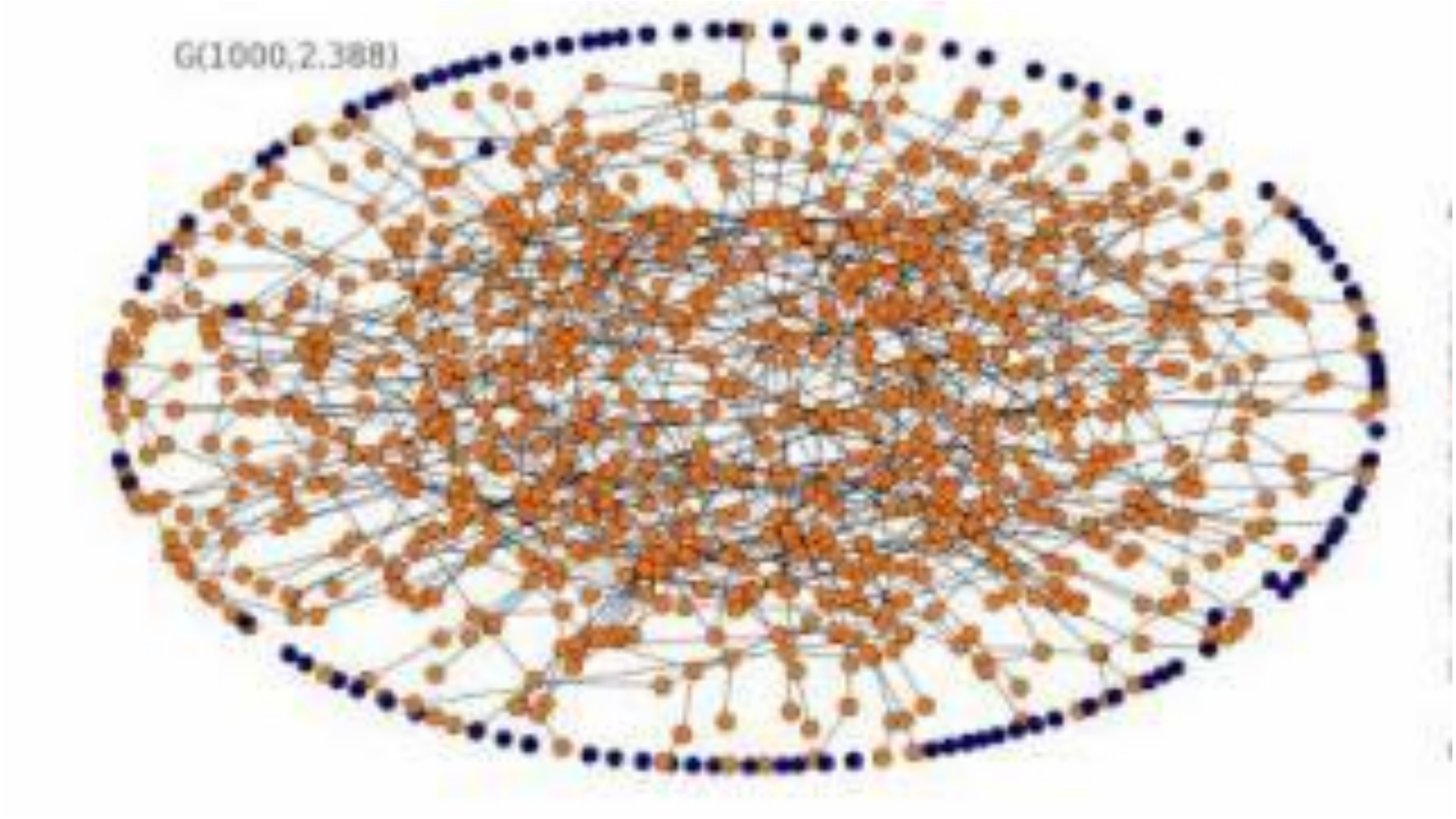
$\Theta(n^{2/3})$

the birth of the giant component

E-R (n, 1/n) for n = 500



Evolution of $(G^{(n)}(t/n), t \geq 0)$ for $n = 1000$

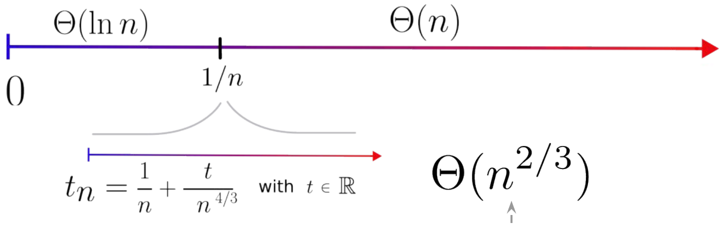


The standard augmented multiplicative coalescent

Critical regime

$$x_i^{(n)} = n^{-2/3}$$ mass of each vertices

$$q_n(t) = n^{1/3} + t$$ rescaled critical time



Metric associated to the AMC (Bhamidi et al. (2014))

$$\mathrm{d}\left((x_i, n_i)_{i\geq 1}, (x'_i, n'_i)_{i\geq 1}\right) = \sum_{n=1}^{\infty}(x_i - x'_i)^2 + \sum_{n=1}^{\infty}|x_i \cdot n_i - x'_i \cdot n'_i|$$

already done
Aldous (1997), Limic (2019)

a bit more complicated...
Aldous (1997), Bhamidi et al. (2014), Broutin and Marckert (2016)

C. and Limic (2023+)

$$\mathrm{d}\left(\left(X_i^{(n)}(q_n(t)), N_i^{(n)}(q_n(t))\right)_{i\geq 1}, \left(\mathcal{X}_i(t), \mathcal{N}_i(t)\right)_{i\geq 1}\right) \xrightarrow[n\to\infty]{\text{law}} 0$$

C. and Limic (2023+)

$$\mathrm{d}\Big(\big(X_i^{(n)}(q_n(t)), N_i^{(n)}(q_n(t))\big)_{i\geq 1}, \big(\mathcal{X}_i(t), \mathcal{N}_i(t)\big)_{i\geq 1}\Big) \xrightarrow[n\to\infty]{\text{law}} 0$$

$$W^t(s) = W(s) - \frac{1}{2}s^2 + t\cdot s$$ Brownian motion with drift

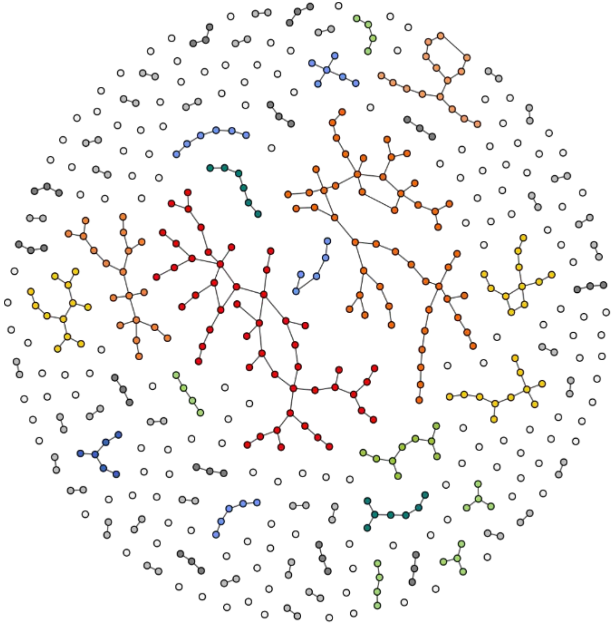
$$B^t(s) = W^t(s) - \inf_{0\leq u\leq s} W^t(u)$$ reflected BM with drift

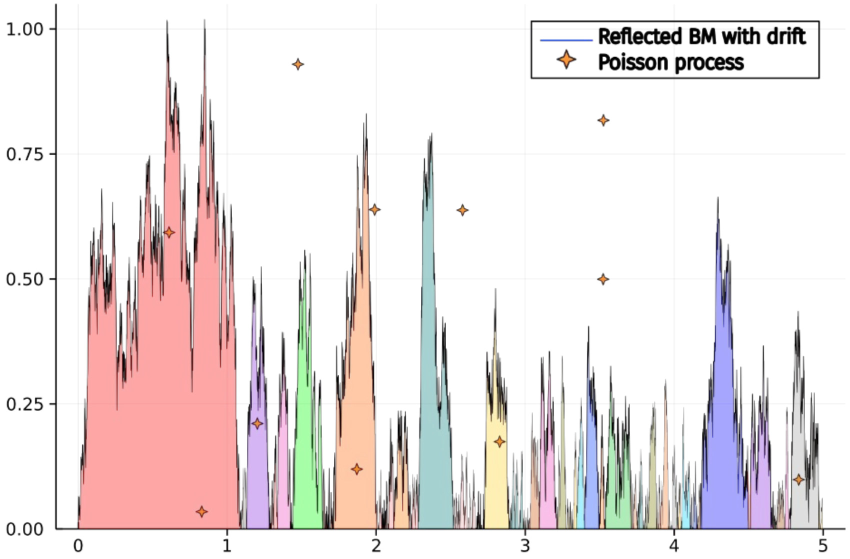
$\Lambda =$ homogeneous Poisson point process on $\mathbb{R}_+^2$

$\mathcal{X}_i(t)$ size of the i-th largest excursion of B^t

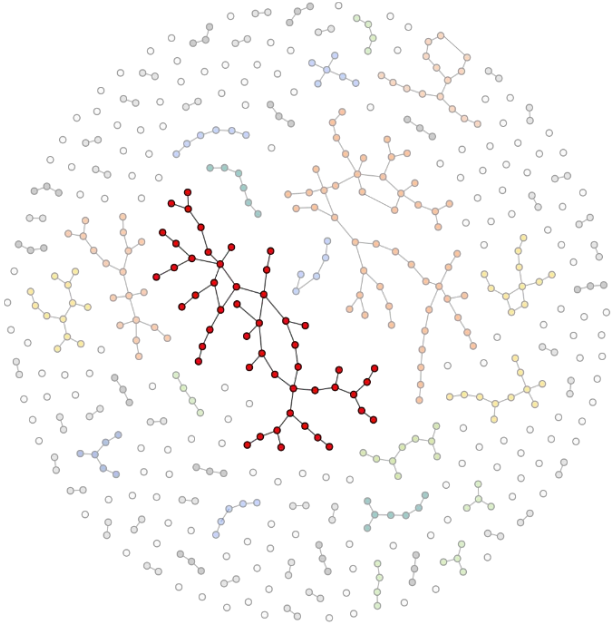
$\mathcal{N}_i(t)$ number of marks of Λ below B^t during the excursion associated to $\mathcal{X}_i(t)$

The same result is valid for the multi-graph and the Erdös-Rényi graph valued processes

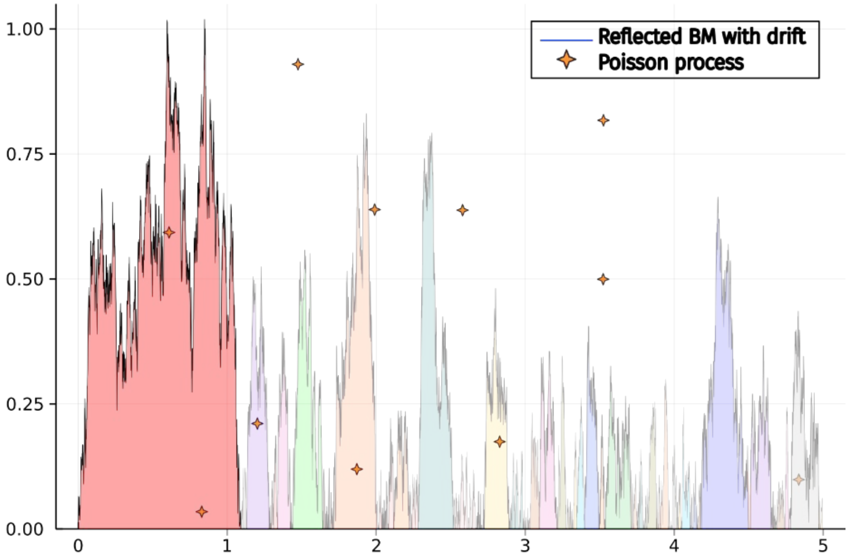




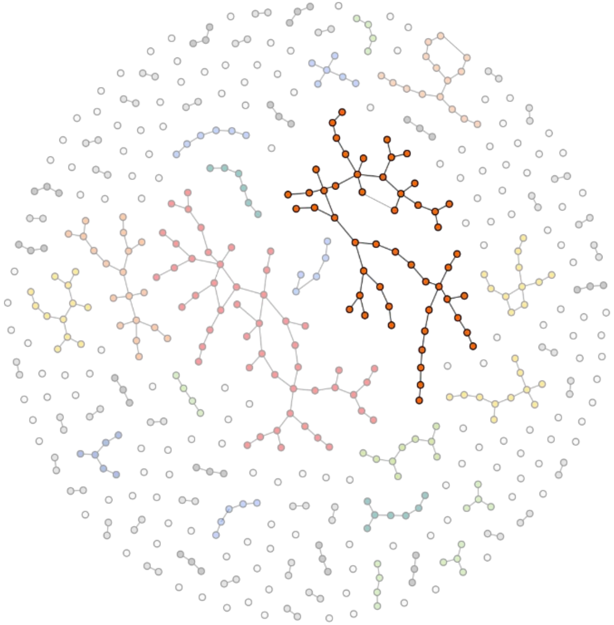
Reflected BM with drift
Poisson process
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5



$n^{-2/3} \cdot$ number of vertices
number of surplus edges



size of the excursions
number of marks below the curve



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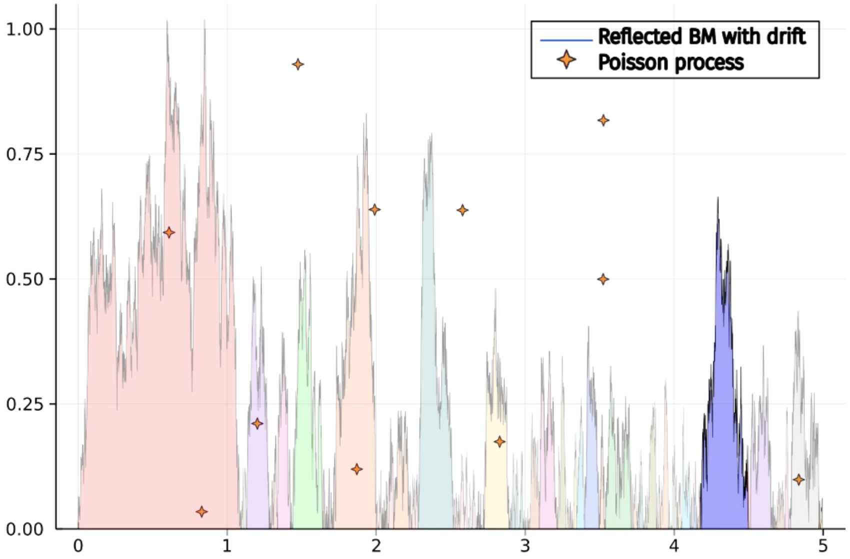

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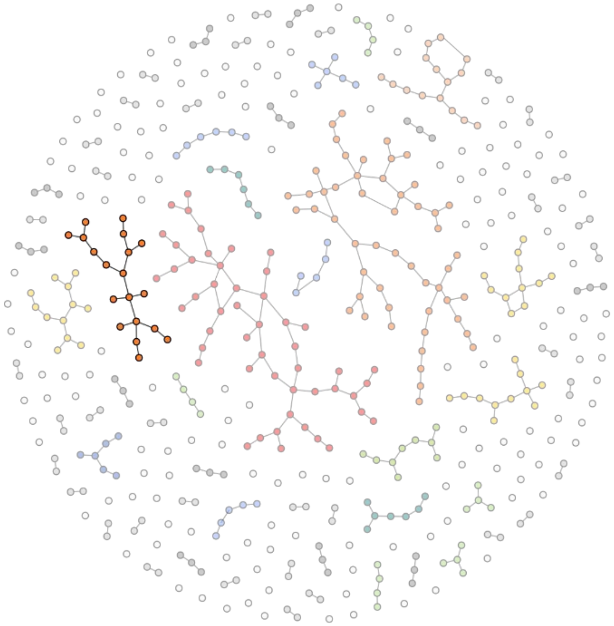

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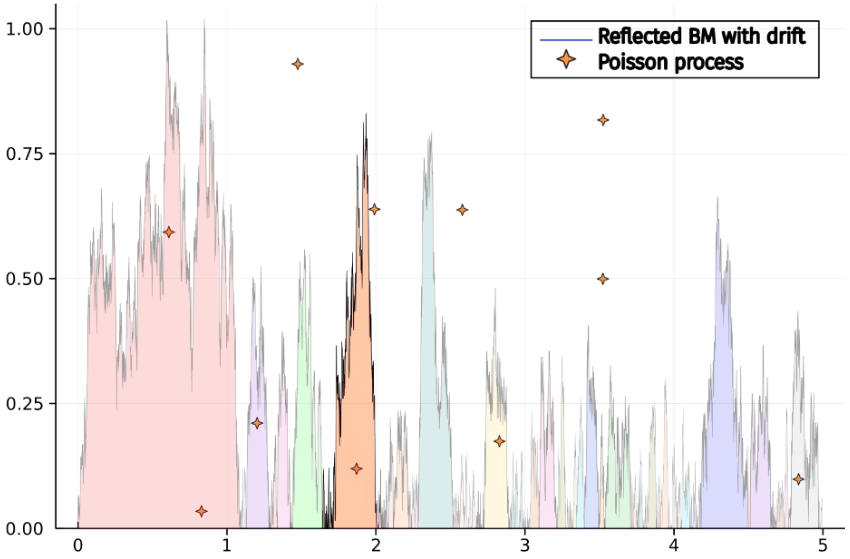

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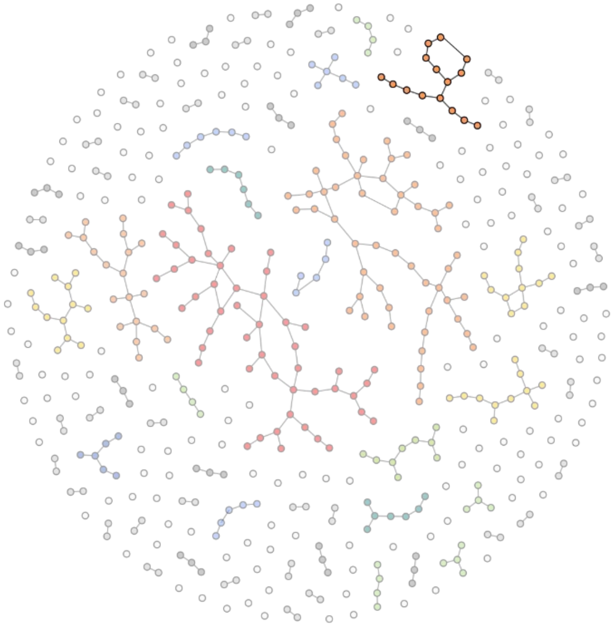

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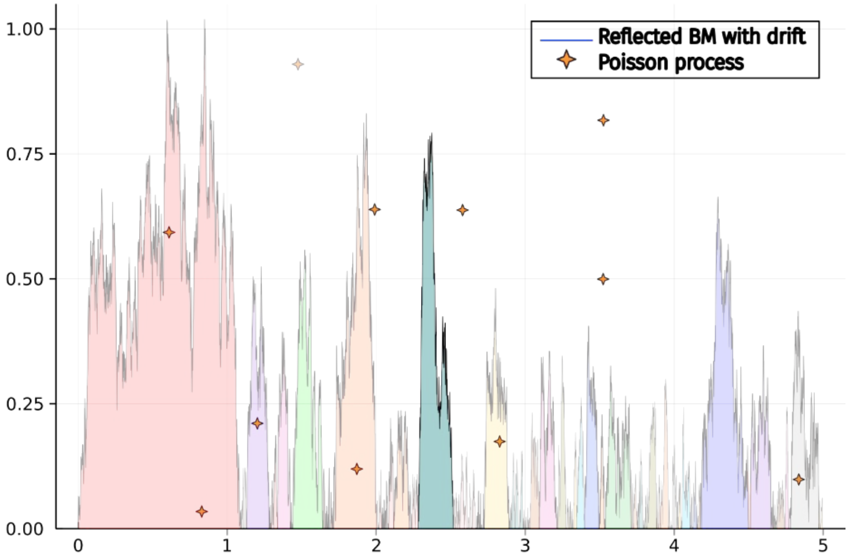
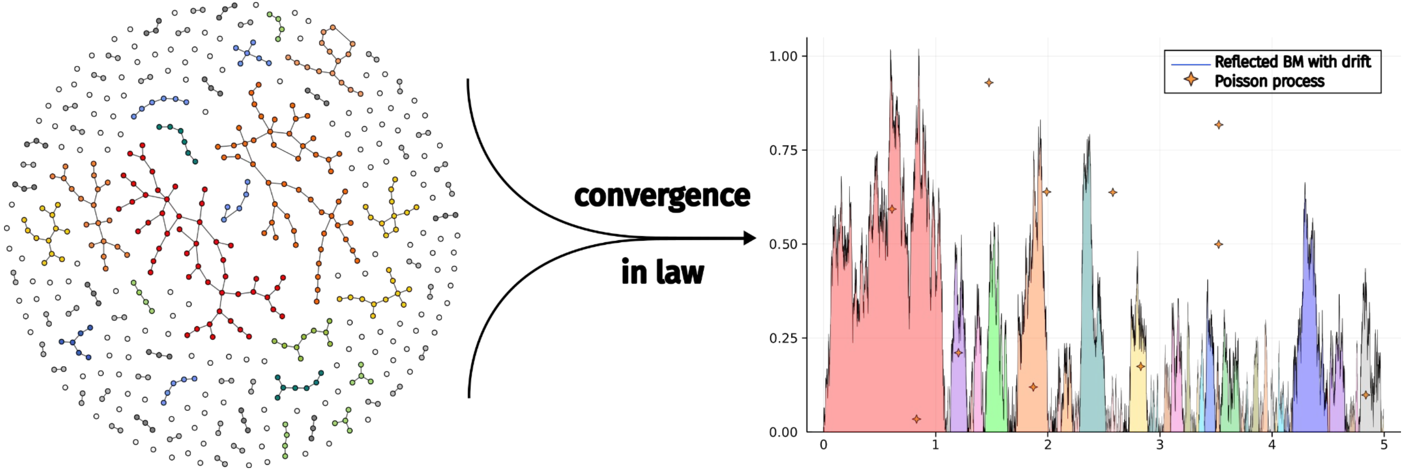


size of the excursions
number of marks below the curve

C. and Limic (2023+)



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− known/expected result (Aldous (1997), Bhamidi et al. (2014), Broutin and Marckert (2016))

+ new methods, simpler and potentially extensible for studying the dynamics of more general (inhomogeneous) models