Processus de croissance-fragmentation en lien avec les excursions browniennes au-dessus des hyperplans

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En Nouvelle-Zélande en collaboration avec le Mexique et l'Anglaterre

Nom du projet: Genealogies of samples of individuals selected at random from stochastic populations: probabilistic structure and applications

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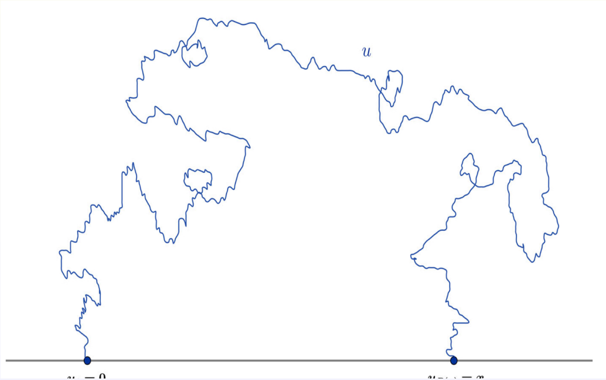

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In other words, the excursion measure $\mathbf{n}_+$ satisfies

$$\mathbf{n}_+(\mathrm{d}x, \mathrm{d}y) = n_+(\mathrm{d}y)\mathbb{P}^{R(y)}(\mathrm{d}x),$$

where n_+ is the excursion measure of positive excursions away from 0 and $\mathbb{P}^{R(y)}$ is the law of a Brownian motion stopped at $R(y)$.

Desintegration over $z = \gamma(R)$

$$\mathbf{n}_+ = \int_{\mathbb{R}} \frac{\mathrm{d}z}{2\pi z^2} \mathbb{P}_z,$$

where $\mathbb{P}_z(\mathrm{d}\gamma)$ can be thought as the law of the brownian excursion in $\mathbb{H}$ between 0 and the end point z.

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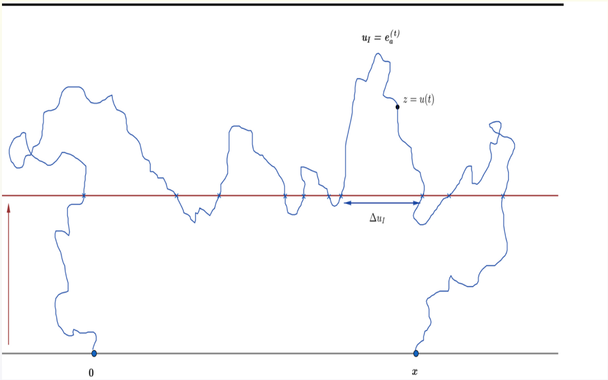
- **iii)** Take a Brownian bridge between 0 and z with duration rz^2 for the real part of γ

Genealogy of subexcursions



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Let $\Delta e_i^{a,+}$ be the size or length of the excursion, i.e. the difference between the endpoint of the excursion and its starting point.

Since both points have the same imaginary part, $(\Delta e_i^{a,+})_{i\geq 1}$ is a collection of real numbers and we suppose that they are ranked in decreasing order of their magnitude.

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The main result in Aïdekon and Da Silva describes the law of the process $(\Delta e_i^{a,+})_{i\geq 1}$ indexed by a in terms of a self-similar growth-fragmentation with two types which is associated with the Cauchy process.

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Our aim is to extend such result for the d-dimensional case.

Spatial self-similar growth-fragmentations

Let $X = (X(t))_{t\geq 0}$ be an isotropic self-similar Markov process with index α. Let $\mathbb{P}_x$ denotes the law of X starting from $x \in \mathbb{R}^d$.

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From the Lamperti-Kiu representation of X (Alili et al. EJP, 2017): there exists a Markov additive process (ξ, Θ) in $\mathbb{R} \times \mathbb{S}^{d-1}$ such that

$$X(t) = \mathrm{e}^{\xi(\varphi(t))}\Theta(\varphi(t)), \quad t \leq I_\varsigma := \int_0^\varsigma \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s,$$

where

$$\varphi(t) := \inf\left\{s > 0,\ \int_0^s \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u > t\right\},$$

and I_ς is the lifetime of X. The converse is also true.

Let $(\xi(t), \Theta(t))_{t\geq 0}$ be a regular Feller process in $\mathbb{R}\times\mathbb{S}^{d-1}$ with probabilities $\mathtt{P}_{x,\theta}$, $x\in\mathbb{R}$, $\theta\in\mathbb{S}^{d-1}$. We say that (ξ,Θ) is a *Markov additive process* (MAP for short) if for every bounded measurable $f:\mathbb{R}\times\mathbb{S}^{d-1}\to\mathbb{R}$, $s,t\geq 0$ and $(x,\theta)\in\mathbb{R}\times\mathbb{S}^{d-1}$,

$$\mathtt{E}_{x,\theta}\left[f(\xi(t+s)-\xi(t),\Theta(t+s))\mathbf{1}_{\{t+s<\varsigma\}}\middle|\mathcal{G}_t\right] = \mathbf{1}_{\{t<\varsigma\}}\mathtt{E}_{0,\Theta(t)}\left[f(\xi(s),\Theta(s))\mathbf{1}_{\{s<\varsigma\}}\right],$$

where $\varsigma := \inf\{t>0,\ \Theta(t)=\dagger\}$.

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In this construction, the cells are not labelled chronologically. However, it still uniquely defines the law $\mathcal{P}_{\mathbf{x}}$ of the cell system $(\mathcal{X}_u(t), u \in \mathbb{U}, t \geq 0)$ started from $\mathbf{x}$.

Finally, we introduce the *(spatial) growth-fragmentation process*

$$\mathbf{X}(t) := \{\{\mathcal{X}_u(t - b_u),\ u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

describing the collection of cells alive at time $t \geq 0$ (b_u and ζ_u denote the birth and life time of u). We define $\mathbf{P}_{\mathbf{x}}$ to be the law of the growth-fragmentation $\mathbf{X}$ started at $\mathbf{x}$.

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This construction can be seen as a *multitype growth-fragmentation* process, where the types correspond to the directions (in the $d = 1$ case, it is the sign). The set of types is therefore the sphere $\mathbb{S}^{d-1}$, which is uncountable.

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The type corresponding to the daughter cell created by the jump $\Delta X(t)$ is, up to time-change,

$$\Theta_\Delta(t) := \frac{\Theta(t^-) - \mathrm{e}^{\Delta\xi(t)}\Theta(t)}{|\Theta(t^-) - \mathrm{e}^{\Delta\xi(t)}\Theta(t)|}.$$

Let

$$\overline{\mathbf{X}}(t) := \{\{(\mathcal{X}_u(t - b_u), |u|),\ u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

$(\mathcal{F}_t, t \geq 0)$ the natural filtration associated with $\mathbf{X}$, and $(\overline{\mathcal{F}}_t, t \geq 0)$ the one associated with $\overline{\mathbf{X}}$.

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Branching property

For any $t \ge 0$, conditionally on $\overline{\mathbf{X}}(t) = \{\{(\mathbf{x}_i, n_i)\}\}$, the process $(\overline{\mathbf{X}}(t+s), s \ge 0)$ is independent of $\overline{\mathcal{F}}_t$ and distributed as

$$\bigsqcup_{i \ge 1} \overline{\mathbf{X}}_i(s) \circ \theta_{n_i},$$

where the $\overline{\mathbf{X}}_i, i \ge 1$, are independent processes distributed as $\overline{\mathbf{X}}$ under $\mathcal{P}_{\mathbf{x}_i}$, θ_n is the shift operator $\{\{(\mathbf{z}_i, k_i), i \ge 1\}\} \circ \theta_n := \{\{(\mathbf{z}_i, k_i + n), i \ge 1\}\}$, and $\sqcup$ denotes union of multisets.

Spine decomposition

Recall that ξ is a Lévy process and assume that its Laplace exponent ψ satisfies $\psi(q) < 0$, then

$$\mathbb{E}_{\mathbf{1}}\left[\sum_{0<t<\zeta} |\Delta X(t)|^q\right] = 1 - \frac{\kappa(q)}{\psi(q)},$$

where $\mathbf{1} = (1, 0, \ldots, 0)$ and κ is computed in terms of the characteristic of the MAP (ξ, Θ).

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We call the function κ the *isotropic cumulant function*. Its roots will lead to martingales for the growth-fragmentation cell system.

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We make the following assumption

(H) *There exists $\omega \geq 0$ such that $\kappa(\omega) = 0$.*

Let $\mathcal{G}_n := \sigma\left(\mathcal{X}_u, |u| \leq n\right), n \geq 0$. The process

$$\mathcal{M}(n) := \sum_{|u|=n+1} |\mathcal{X}_u(0)|^{\omega}, \quad n \geq 0,$$

is a $(\mathcal{G}_n)_{n\geq 0}$-martingale under $\mathcal{P}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$.

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For all $G_n \in \mathcal{G}_n$

$$\hat{\mathcal{P}}_{\mathbf{x}}(G_n) := |\mathbf{x}|^{-\omega} \mathcal{E}_{\mathbf{x}}\left[\mathcal{M}(n) \mathbf{1}_{G_n}\right].$$

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It describes the law of a new cell system $(\mathcal{X}_u)_{u\in\mathbb{U}}$ together with an infinite distinguished *ray*, or *leaf*, $\mathcal{L} \in \partial\mathbb{U} = \mathbb{N}^{\mathbb{N}}$.

The law of the particular leaf $\mathcal{L}$ under $\hat{\mathcal{P}}_z$ is chosen so that, for all $n \geq 0$ and all $u \in \mathbb{U}$ such that $|u| = n+1$

$$\hat{\mathcal{P}}_{\mathbf{x}} \left(\mathcal{L}(n+1) = u \,\middle|\, \mathcal{G}_n \right) := \frac{|\mathcal{X}_u(0)|^{\omega}}{\mathcal{M}(n)}, \tag{1}$$

In words, to define the next generation of the spine, we select one of its jumps proportionally to its size to the power ω.

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Tagged cell: is the cell associated with the distinguished leaf $\mathcal{L}$. For any $\ell \in \partial\mathbb{U}$, $\ell(n)$ denotes the ancestor of ℓ at generation n. More precisely, set $b_\ell = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial\mathbb{U}$. Then, define $\hat{\mathcal{X}}$ by $\hat{\mathcal{X}}(t) := \partial$ if $t \geq b_{\mathcal{L}}$ and

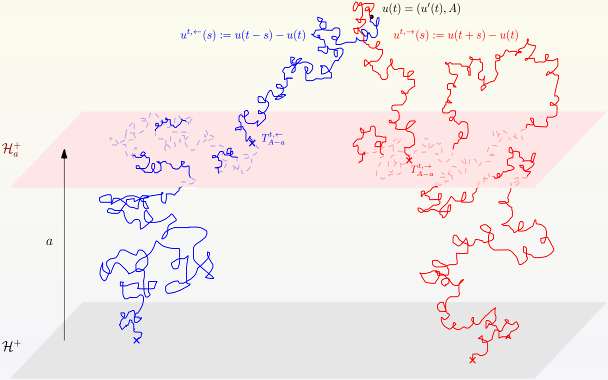
$$\hat{\mathcal{X}}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}), \quad t < b_{\mathcal{L}}.$$

where n_t is the unique integer n such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$.

Under $\hat{\mathcal{P}}_{\mathbf{x}}$, $\hat{\mathcal{X}}$ is a self-similar Markov process with values in $\mathbb{R}^d$ and index α. Besides, $\hat{\mathcal{X}}$ is isotropic, and the ordinate $\hat{\xi}$ is a Lévy process with Laplace exponent $\hat{\psi}(q) = \kappa(\omega + q)$.

Excursions from hyperplanes

Let $d \geq 3$ and $\mathcal{H} = \{x_d = 0\}$.



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(i) if $\tau_s - \tau_{s^-} > 0$, then

$$\mathbf{e}_s : r \mapsto \left(B^{d-1}_{r+\tau_{s^-}} - B^{d-1}_{\tau_{s^-}}, Z_{r+\tau_{s^-}} \right), \quad r \leq \tau_s - \tau_{s^-},$$

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(ii) if $\tau_s - \tau_{s^-} = 0$, then $\mathbf{e}_s = \partial$,

where ∂ is some cemetery state.

The excursion process $(\mathbf{e}_s, s > 0)$ is a Poisson point process with intensity measure

$$\mathbf{n}(\mathrm{d}u', \mathrm{d}z) := n(\mathrm{d}z)\mathbb{P}\Big((B^{d-1})^{R(z)} \in \mathrm{d}u'\Big),$$

where n denotes the one-dimensional Itô excursion measure and for any process X, and any time T, $X^T := (X_t,\, t \in [0, T])$.

Desintegration

The following disintegration formula holds:

$$\mathbf{n}_+ = \int_{\mathbb{R}^{d-1}\setminus\{0\}} \mathrm{d}\mathbf{x}\, \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}|\mathbf{x}|^d} \cdot \gamma_{\mathbf{x}},$$

where $\gamma_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^{d-1} \setminus \{0\}$, are probability measures. In addition, for all $\mathbf{x} \in \mathbb{R}^{d-1} \setminus \{0\}$,

$$\gamma_{\mathbf{x}} = \int_0^\infty \mathrm{d}r \frac{\mathrm{e}^{-\frac{1}{2r}}}{2^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right) r^{\frac{d}{2}+1}} \mathbb{P}^{0\to\mathbf{x}}_{r|\mathbf{x}|^2} \otimes \Pi_{r|\mathbf{x}|^2},$$

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where for $r \geq 0$ and $\mathbf{x} \in \mathbb{R}^{d-1}$, Π_r is the law of a Bessel bridge from 0 to 0 over $[0, r]$, and $\mathbb{P}^{0\to\mathbf{x}}_r$ is the law of a $(d-1)$-dimensional Brownian bridge from 0 to $\mathbf{x}$ with duration r.

Slicing excursions

Define the superlevel set

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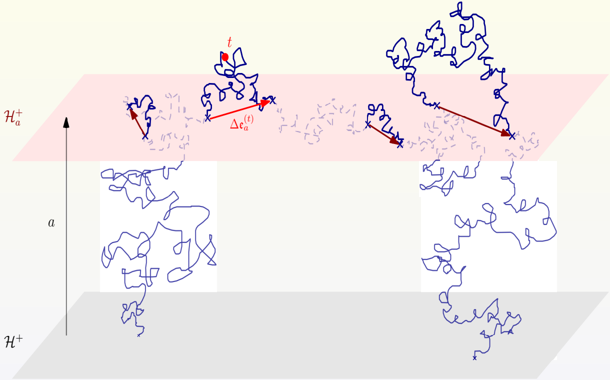
This is a countable (possibly empty) union of disjoint open intervals, and for any such interval $I = (i_-, i_+)$, we write

$$u_I(s) := u(i_- + s) - u(i_-), \qquad 0 \leq s \leq i_+ - i_-,$$

for the restriction of u to I, and $\Delta u_I := u(i_+) - u(i_-)$.

Observe that Δu_I is a vector in the hyperplane $\mathcal{H}_a := \{x_d = a\}$, which we call the *size* or *length* of the excursion u_I. If $0 \leq t \leq R(u)$, we denote by $e_a^{(t)}$ the excursion u_I corresponding to the unique interval I which contains t.

Moreover, we define $\mathcal{H}_a^+$ as the set of excursions above $\mathcal{H}_a$ corresponding to the previous partition of $\mathcal{I}(a)$.



The red arrows indicate the *size* of the sub-excursions, counted with respect to the orientation of u

Let

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we prove that $\mathbf{Z}$ enjoys a branching property akin to the temporal branching property of spatial growth-fragmentation and actually under $\gamma_{\mathbf{x}}$, we observe that $\mathbf{Z}$ is a *spatial growth-fragmentation process*.

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We construct a temporal martingale in terms of excursions in $\mathcal{H}_a^+$ which is a temporal version of the martingale for spatial growth-fragmentation. Using this, we determine the law of the spine.

The spine is described as a $(d-1)$-dimensional Brownian motion taken at the hitting times of another independent linear Brownian motion, and hence is a $(d-1)$-dimensional isotropic Cauchy process.

Merci beaucoup!