Regularity of the last-passage percolation time constant on complete directed acyclic graphs

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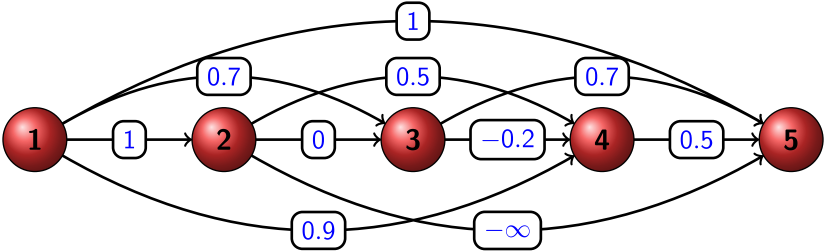
Overview

1. The last-passage percolation model on complete graphs
2. A motivating example: parallel computing with precedence constraints
3. Main results
4. Some elements of proof

Last-passage percolation on complete graphs

- Set of vertices: $V_n = \{1, \ldots, n\}$
- Set of **directed edges**: $E_n = \{\,(i,j) \mid 1 \leq i < j \leq n\,\}$
- ν is a probability distribution on $\{-\infty\} \cup \mathbb{R}$
- **Random weight** $X_{i,j}$ on each edge $(i,j) \in E_n$, where $(X_{i,j})_{i<j}$ are i.i.d. with distribution ν



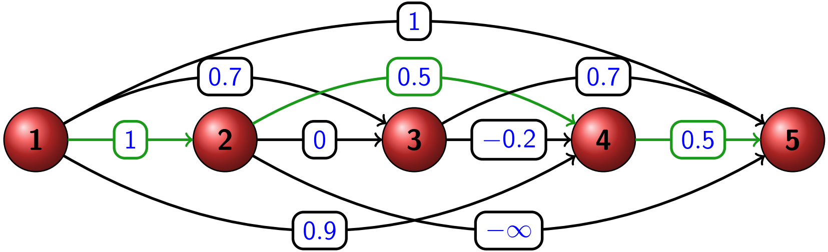
Last-passage percolation on complete graphs

- We consider directed paths in the graph.
- The weight of a path is the sum of the weights of its edges.
- W_n the **maximal weight of paths starting from** 1 **and ending at** n.

Property

There exists a deterministic constant $C(\nu) \in [0, +\infty]$ *called* **time constant** *such that*

$$\frac{W_n}{n} \xrightarrow[n\to\infty]{a.s,\ L^1} C(\nu).$$

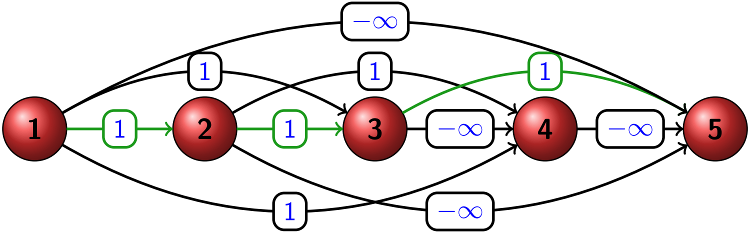


A particular case: the longest path in Barak-Erdős graphs

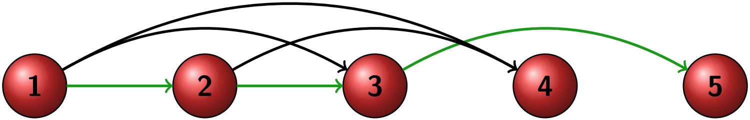
Remark

In terms of a heaviest path, having $X_{i,j} = -\infty$ is equivalent to removing the edge (i, j) from the graph.

Heaviest path



Longest path



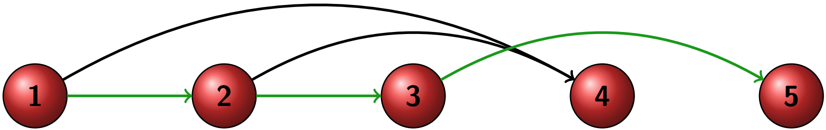
A motivating example for Barak-Erdős graphs

Processing time for parallel computing with precedence constraints

- n tasks to process, infinite number of processors
- processing time 1 for each task
- **Precedence constraints** that must be satisfied during processing given by a **task graph**: task i must be processed before task j $\Longleftrightarrow$ (i, j) is in the task graph.

A motivating example for Barak-Erdős graphs

Example of task graph with $n = 5$ vertices:



Processing time = number of vertices in a path of maximal length

What if we consider a random task graph?

A motivating example for Barak-Erdős graphs

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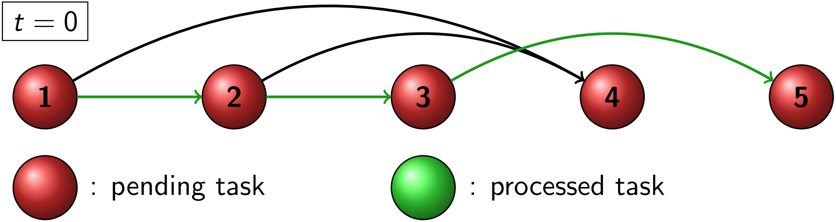

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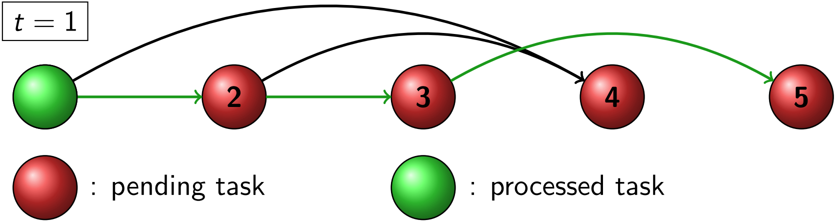

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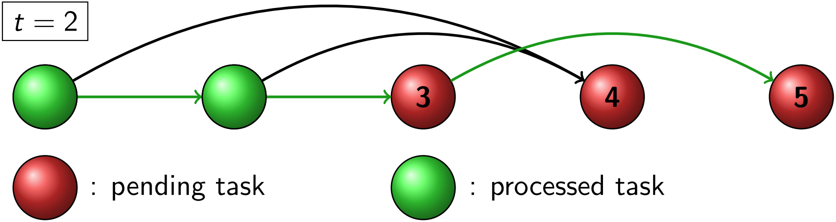

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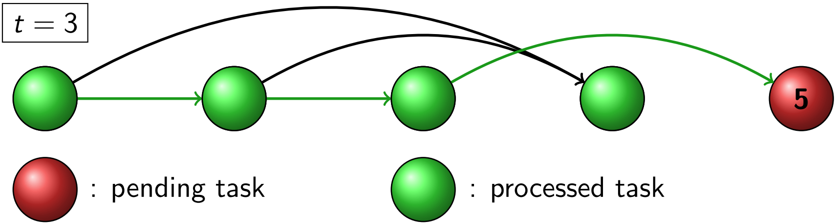

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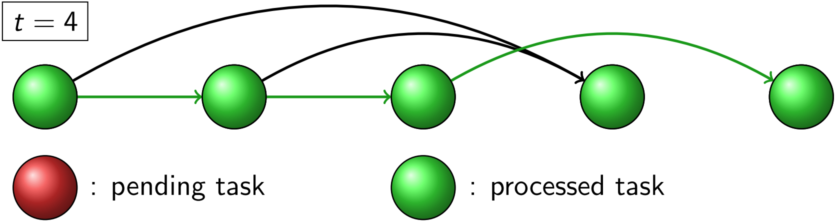


Processing time = number of vertices in a path of maximal length

What if we consider a random task graph?

Previous results on Barak-Erdős graphs

Let ν_p be the distribution of a random variable equal to:

- 1 with probability p,
- $-\infty$ with probability $1-p$.

For i.i.d. weights $(X_{i,j})_{i<j}$ with distribution ν_p,

$$\frac{W_n}{n} \xrightarrow[n\to\infty]{\text{a.s, } L^1} C(\nu_p).$$

Theorem (Mallein, Ramassamy; 19', 21')

- *$p \mapsto C(\nu_p)$ is analytic on $(0,1]$.*
- *The Taylor expansion of $C(\nu_p)$ at $p=1$ has integer coefficients.*
- *$C(\nu_p) = ep + \frac{\pi^2 ep}{2 log(p)^2}(1+o(1))$ as $p \to 0$.*

Main results

- Fix k real numbers $-\infty \leq a_k < \cdots < a_1$
- For any positive real numbers $(p_1, \ldots, p_k)$ such that $p_1 + \cdots + p_k = 1$, consider the probability distribution

$$\nu_{(p_1,\ldots,p_k)} := \sum_{i=1}^{k} p_i \delta_{a_i}.$$

Theorem (T. 23')

The map $(p_1, \ldots, p_k) \mapsto C(\nu_{p_1,\ldots,p_k})$ is **analytic** *on the set $\{\, (p_1, \ldots, p_k) \in [0,1]^k \mid p_1 + \cdots + p_k = 1,\ p_1 > 0 \,\}$.*

Main results

- For ν a probability distribution with upper-bounded support, set $M_\nu := \inf\{\, t \in \mathbb{R} \mid \nu([t, +\infty]) = 0 \,\}$ the **essential supremum** of ν.
- For two probability distributions ν and ν', set

$$d(\nu, \nu') = \max(d_{LP}(\nu, \nu'), |M_\nu - M_{\nu'}|),$$

where d_{LP} is the Lévy-Prokhorov metric.

Theorem (T. 23')

$\nu \mapsto C(\nu)$ is **continuous** *for the metric d on the set of probability measures ν with upper-bounded support.*

Main results

Let $\nu_{p,m}$ be the distribution of a random variable equal to:

- 1 with probability p,
- m with probability $1 - p$.

Theorem (T. 23')

For any real number $m > 0$, $p \mapsto C(\nu_{p,m})$ is a **rational function** *on $[0, 1]$.*

Main results

- Consider ν_1 and ν_2 two probability distributions.
- ν_1 is **stochastically dominated** by ν_2 when for all $t \in \mathbb{R}$,

$$\nu_1((t, +\infty)) \leq \nu_2((t, +\infty)).$$

Theorem (T. 23')

$\nu \mapsto C(\nu)$ is **strictly increasing** *for the stochastic order on the set of distributions with positive finite essential supremum.*

Elements of proof

Let μ be a probability measure on $[-\infty, 1)$.

Let $\nu_{p,\mu} = p\delta_1 + (1-p)\mu$ be the distribution of a random variable equal to:

- 1 with probability p,
- a random variable with distribution μ with probability $1-p$.

Theorem (T. 23')

$p \mapsto C(\nu_{p,\mu})$ *is* **analytic** *on* $(0, 1]$.

Elements of proof: coupling with a particle system

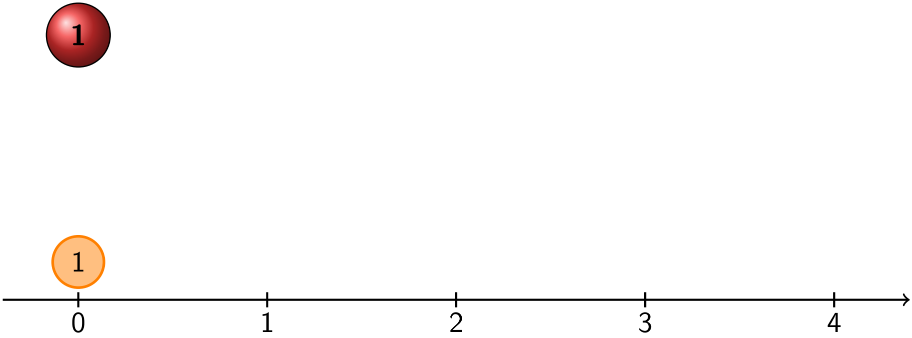
- Coupling last-passage percolation with a particle system called the **max-growth system** (Foss, Konstantopoulos, Mallein, Ramassamy, 23')
- We construct the graph iteratively adding one vertex at the time:

Maximal weight of a path starting at 1 and ending at n

=

Position of the new particle at time n

Elements of proof: coupling with a particle system



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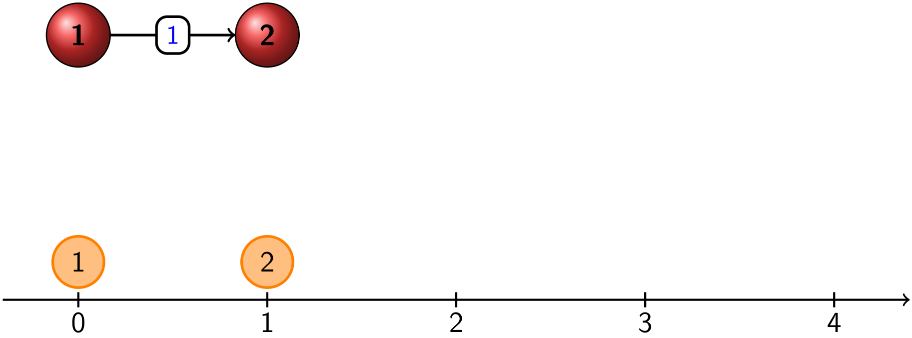

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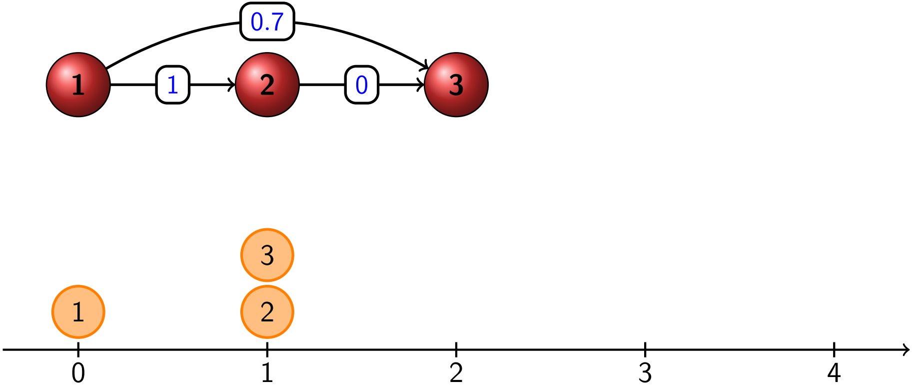

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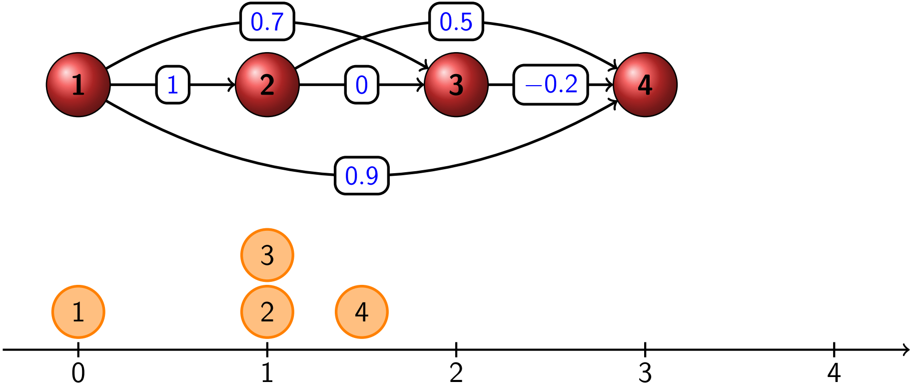

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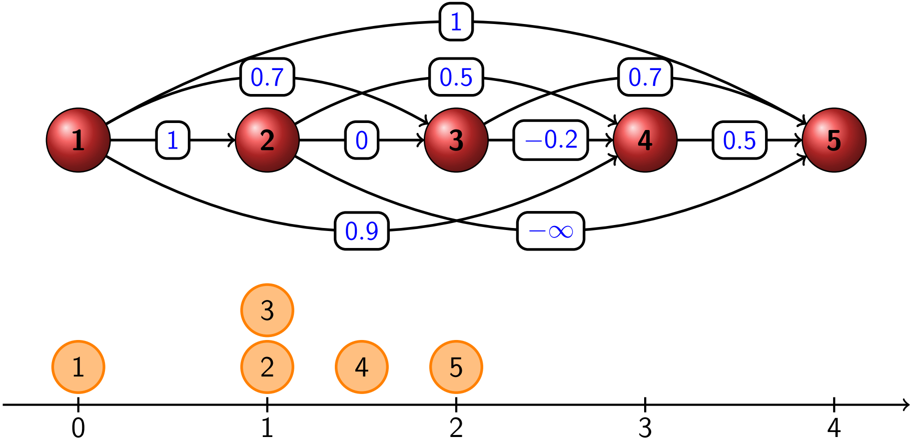

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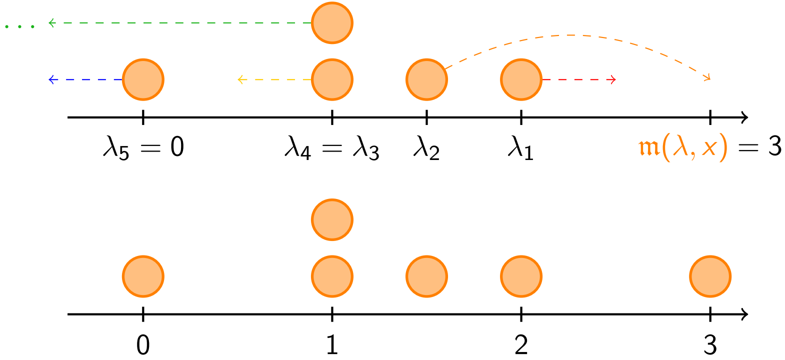
Position of the new particle at time n

Elements of proof: dynamics of the Max Growth System

- A particle configuration: $\lambda = \sum_{i=1}^{N} \delta_{\lambda_i}$ where $\lambda_N \leq \cdots \leq \lambda_1$
- A sequence of weights $X = (X_1, X_2, \ldots, X_N) \in (\{-\infty\} \cup \mathbb{R})^N$

Illustration of the dynamics with $N = 5$ and $X = (0.5, 1.5, -0.5, -\infty, -1)$.



Position of the new particle: $\mathfrak{m}(\lambda, X) := \max_{1 \leq i \leq N}(\lambda_i + X_i)$.

Dynamics of the Max Growth System

- Consider $(X_i^{(n)})_{i,n\in\mathbb{N}^*}$ i.i.d. random variables with distribution ν.
- We start at time zero with a single particle at time zero: $\lambda^{(0)} = \delta_0$.
- For all $n \in \mathbb{N}^*$, we obtain the configuration at time n from the configuration at time $n-1$ by using the sequence of i.i.d. weights $X^{(n)} = (X_i^{(n)})_{1\leq i\leq n}$

Elements of proof: renovation events

- Assume that the support of ν is upper-bounded by 1.
- Notice that if $X_i^{(n)} = 1$, the position of the n-th particle does not depend on $(X_j^{(n)})_{j>i}$.

$X^{(1)} = (1)$
$X^{(2)} = (-1, 1)$
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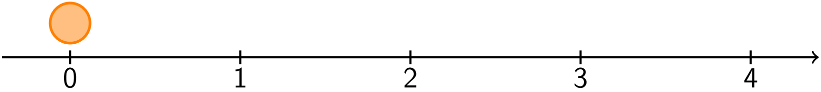

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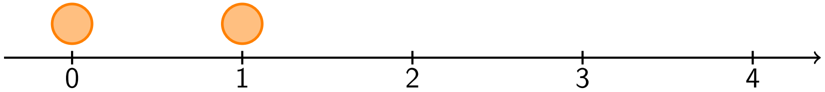

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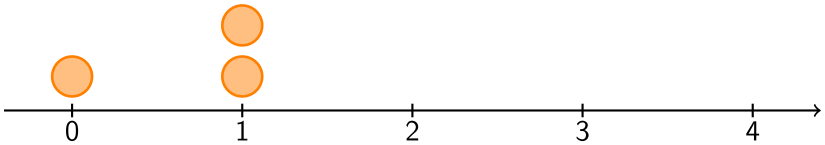

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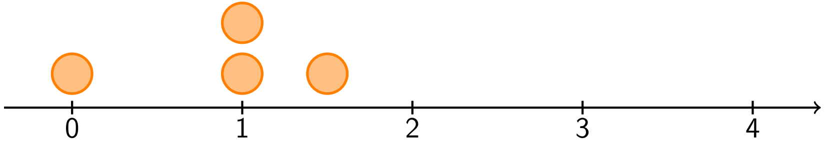

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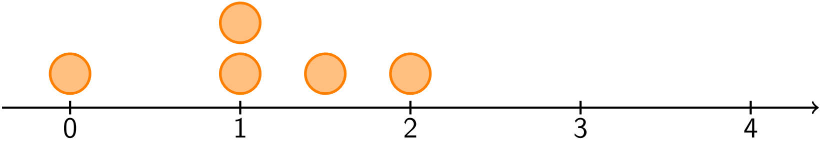

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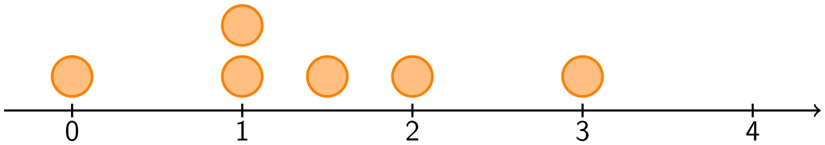

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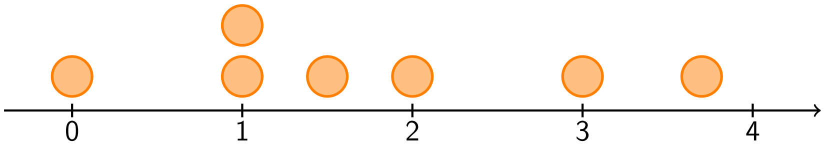

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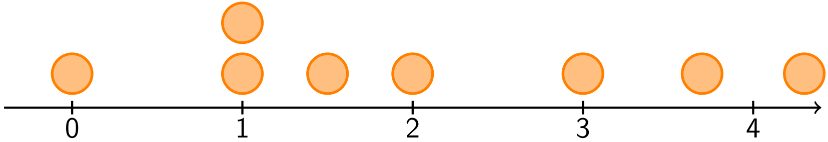

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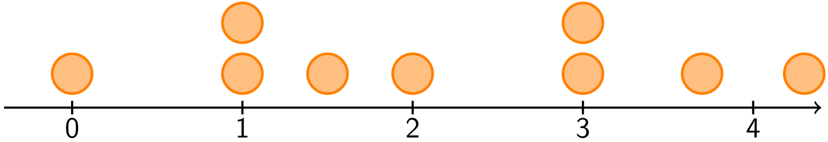

Elements of proof: renovation events

- A **renovation time** is a time n at which the positions of all the future new particles in the system do not depend on the positions of the old particles.
- Sufficient condition for n to be a renovation time: there is at least a 1 in the first $k+1$ weights of the sequence at each time $n+k$ for all $k \geq 0$

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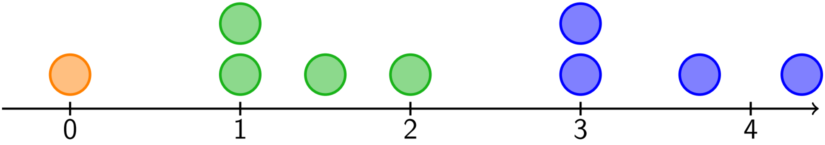

Elements of proof: renovation events

- Set $\mathcal{T} = \{ T_1 < T_2 < \dots \} \subseteq \mathbb{N}^*$ the set of all renovation times.
- By ergodicity:

$$\lim_{N \to \infty} \frac{W_n}{N} = C(\nu) = \frac{\mathbb{E}[W_{T_1,T_2}]}{\mathbb{E}[T_2 - T_1]}$$

- To prove the analyticity result, it suffices to prove that $\mathbb{E}[r^{T_2 - T_1}]$ is finite for some $r > 1$.

Open questions

- **Barak-Erdős case:** For $\nu = p\delta_1 + (1-p)\delta_{-\infty}$, the Taylor expansion of $C(\nu)$ in $q = 1-p$ at $q = 0$ is (On-line Encyclopedia of Integer Sequences: A321309)

  $$1 - q + q^2 - 3q^3 - 7q^4 + 15q^5 - 29q^6 + 54q^7 - 102q^8 \dots$$

  Is there a combinatorial interpretation of those coefficients ?
- Same question for $\nu = p\delta_1 + (1-p)\delta_{-k}$ with $k \in \mathbb{N}^*$? Can we get its asymptotics at $p = 0$?

Thank you!