Parameters estimation of a Threshold CIR process

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Summary

Introduction

The Cox–Ingersoll–Ross (CIR) model is a stochastic process that exhibits mean-reverting behaviour.

- In finance, the CIR is used to describes the evolution of interest rates.
- In biology, the CIR can be used to model population dynamics [Bansaye & Méléard, 2015].

Threhsold Diffusion:

- Threshold autoregressive (TAR) models in discrete time were introduced in the early 1980s.
- Many applications in Finance, Physics and meteorology.

The Threshold CIR follows the CIR dynamics when above or below a fixed level, yet at this level (threshold) its coefficients can be discontinuous.

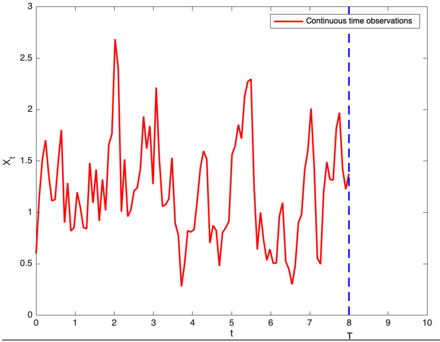
Content

Goal: Estimate the parameters of a TCIR process by using the observations of a single trajectory.

- MLE drift estimation.

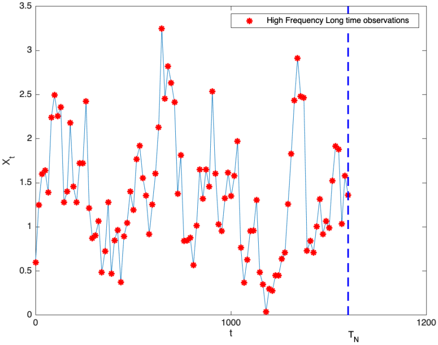
$\rightarrow$ Study of the asymptotic behavior of the estimators (Consistency, Asymptotic Normality).

- Continuous time observations.
- High frequency and long time observations.



Continuous time observations:

We observe the process on the interval $[0, T]$ with $T \in (0, \infty)$.



High Frequency in Long time observations:

We observe the process on the discrete time grid $0 < t_0 < \cdots < t_N = T$, for $N \in \mathbb{N}$ and $T \in (0, \infty)$.

We work on the probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Cox–Ingersoll–Ross (CIR) process

Let $(X_t)_{t \geq 0}$, the process solution of

$$\begin{cases} dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t} dB_t, \quad t > 0, \\ X_0 = x_0, \end{cases} \tag{1}$$

with $(a, b, \sigma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^\star$, $(B_t)_{t \geq 0}$ a standard Brownian motion.

Strong existence of a unique solution to (1) follows from the classical results in 1-D.

Property

- For all $t \geq 0$, the process $(X_t)_{t \geq 0}$ is positif (Comparison Theorem).
- If $a \leq \sigma$, $\{0\}$ instantaneously reflecting.
- If $a > \sigma$, the state space of the process is $]0, +\infty[$.

Literature

In [Alaya & Kebaier, 2013], the authors study the Maximum Likelihood Estimator for the parameters (a, b).

Likelihood ration/Girsanov Weight

Suppose that the one dimensional diffusion process $(X_t^\theta)_{t\geq 0}$ satisfies

$$dX_t^\theta = b(\theta, X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0.$$

- Continuous observations $(X_t^{\theta_0})_{t\in[0,T]}$.
- $\theta_0 \in \Theta$ the parameter to be estimated.

For any $\theta \in \Theta$, the Likelihood ratio is given by

$$G_T^{\theta,\theta_0} := \frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\theta_0,T}} = \exp\left(\int_0^T \frac{b(\theta,X_s)-b(\theta_0,X_s)}{\sigma^2(X_s)} dX_s - \frac{1}{2}\int_0^T \frac{b^2(\theta,X_s)-b^2(\theta_0,X_s)}{\sigma^2(X_s)} ds \right).$$

For the CIR process the Likelihood ratio, evaluated at time T is given by

$$G_T(\theta) = G_T(a,b) := G_T^{\theta,\theta_0} = \exp\left(\frac{1}{2\sigma} \int_0^T \frac{a-bX_s}{X_s} dX_s - \frac{1}{4\sigma} \int_0^T \frac{(a-bX_s)^2}{X_s} ds \right).$$

$$(\hat{a}_T, \hat{b}_T) = \underset{(a,b)\in\Theta}{\text{Argmax}}\ G_T(a,b) \text{ (Explicitly computable)}.$$

Remark

The estimators and the Likelihood ratio is well defined iff $a > \sigma$ *i.e.*

$$\mathbb{P}_\theta\left(\int_0^T \frac{ds}{X_s} < \infty \right) = 1.$$

So $\Theta \in \{(a,b) \text{ s.a. } a > \sigma\}$.

Study of the asymptotic behavior in the ergodic case ($b > 0$) and non ergodic case ($b \leq 0$).

- in long time from continuous time observations.
- in long time from high frequency observations (for any $x_0 \in \mathbb{R}^+$).

Threshold Cox-Ingersoll-Ross (TCIR) model

Let $(X_t)_{t\geq 0}$, the process solution of

$$\begin{cases} dX_t = (a_r(X_t) - b_r(X_t)X_t)dt + \sqrt{2\sigma_r(X_t)X_t}dB_t, & t > 0, \\ X_0 = x_0. \end{cases} \tag{2}$$

$$a_r(x) = \begin{cases} a_+ \in \mathbb{R} & \text{si } x \geq r, \\ a_- \in \mathbb{R}^+ & \text{si } x < r, \end{cases} \quad b_r(x) = \begin{cases} b_+ \in \mathbb{R} & \text{si } x \geq r, \\ b_- \in \mathbb{R} & \text{si } x < r, \end{cases}$$

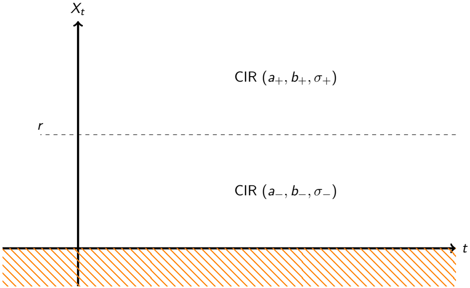
$$\sigma_r(x) = \begin{cases} \sigma_+ > 0 & \text{si } x \geq r, \\ \sigma_- > 0 & \text{si } x < r. \end{cases} \tag{3}$$

with $r > 0$ et $x_0 > 0$.

Issue: Drift and volatility does not fit into the classical hypotheses of strong existence and unicity of a dimensional EDS.

$\rightarrow$ Strong existence of a unique solution to (1) follows from the results of [Mazzonetto & Nieto, in progress].

Property of the TCIR



Separately on (r, ∞) and $(0, r)$, the process follows the CIR dynamics.

Property

- For all $t \geq 0$, the process $(X_t)_{t \geq 0}$ is positif (Comparison Theorem [Mazzonetto & Nieto, in progress]).
- If $a_- \leq \sigma_-$, $\{0\}$ instantaneously reflecting.
- If $a_- > \sigma_-$, the state space of the process is $]0, +\infty[$.

Regime of the process

We denote the scale function S and speed measure m:

$$S'(x) = \exp\left(\int^x \frac{(a_r(y) - b_r(y)y)}{\sigma_r(y)y} dy\right) \text{ and } m(x) = \frac{1}{\sigma_r(x)xS'(x)}.$$

Long time behavior depends on the parameters:

	Null recurrent	Positive Recurrent (Ergodic)	Transience
$\{0\}$ Absorbing point ($a_- = 0$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $a_- \leq 0]$	$\emptyset$	$[b_+ < 0$ or $a_+ \geq \sigma_+]$
$\{0\}$ Instantaneous reflecting point ($a_- \leq \sigma_-$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $0 < a_- \leq \sigma_-]$.	$[[b_+ > 0$ and $a_+ \in \mathbb{R}]$ or $[b_+ = 0$ and $a_+ \in \mathbb{R}^\star_-]]$ and $[b_- \in \mathbb{R}$ and $0 < a_- \leq \sigma_-]$.	$[b_+ < 0$ or $a_+ \geq \sigma_+]$
$\{0\}$ is not reached ($a_- > \sigma_-$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $a_- > \sigma_-]$	$[[b_+ > 0$ and $a_+ \in \mathbb{R}]$ or $[b_+ = 0$ and $a_+ \in \mathbb{R}^\star_-]]$ and $[b_- \in \mathbb{R}$ and $a_- > \sigma_-]$.	$[b_+ < 0$ or $a_+ \geq \sigma_+]$

In the positive recurrent case, the stationary measure μ, is given by

$$\mu(dx) = \frac{m(x)}{\int_0^{+\infty} m(y)dy} dx.$$

Likelihood TCIR

We denote $\theta = (a_+, b_+, a_-, b_-)$, the likelihood ratio evaluated at time T is given by

$$\mathrm{G}_T(\theta) = \exp\left(\int_0^T \frac{a_r(X_s) - b_r(X_s)X_s}{2\sigma_r(X_s)X_s} dX_s - \frac{1}{4}\int_0^T \frac{(a_r(X_s) - b_r(X_s)X_s)^2}{\sigma_r(X_s)X_s} ds\right).$$

Remark

The estimators and the Likelihood is well defined iff $a_- > \sigma_-$ *i.e.*

$$\mathbb{P}_\theta\left(\int_0^T \frac{1_{X_s \leq r} ds}{X_s} < \infty\right) = 1.$$

For the estimation problem, we suppose r and $\sigma_\pm$ to be known.

Drift estimation from continuous time observations

Imagine that we observe $(X_t)_{t\in[0,T]}$ in continuous time. For $T \in (0,\infty)$ and $m = -1, 0, 1$, we define

$$\mathcal{Q}_T^{\pm,m} = \int_0^T X_s^m 1_{\{\pm(X_s - r)\geq 0\}} ds \quad \text{and} \quad \mathcal{M}_T^{\pm,m} = \int_0^T X_s^m 1_{\{\pm(X_s - r)\geq 0\}} dX_s.$$

Let $\theta_0 = (a_0^{\pm}, b_0^{\pm})$, the parameter to be estimated and $\theta \in \Theta \subset \mathbb{R}^4$. We denote

$$(\alpha_T^{\pm}, \beta_T^{\pm}) = \underset{\theta\in\Theta}{\text{Argmax}}\ G_T(\theta).$$

Estimators

For every $T \in (0,\infty)$ the MLE are given by

$$\alpha_T^{\pm} = \frac{\mathcal{M}_T^{\pm,-1}\mathcal{Q}_T^{\pm,1} - \mathcal{M}_T^{\pm,0}\mathcal{Q}_T^{\pm,0}}{\mathcal{Q}_T^{\pm,1}\mathcal{Q}_T^{\pm,-1} - (\mathcal{Q}_T^{\pm,0})^2} \text{ and } \beta_T^{\pm} = \frac{\mathcal{M}_T^{\pm,-1}\mathcal{Q}_T^{\pm,0} - \mathcal{Q}_T^{\pm,-1}\mathcal{M}_T^{\pm,0}}{\mathcal{Q}_T^{\pm,1}\mathcal{Q}_T^{\pm,-1} - (\mathcal{Q}_T^{\pm,0})^2},$$

Theorem MLE: long-time behavior (positive recurrent cases)

Let $\pm \in \{+,-\}$. It holds that

- $\frac{1}{T}\mathcal{Q}_T^{\pm,m} \xrightarrow[T\to\infty]{\text{a.s.}} \mathcal{Q}_\infty^{\pm,m} \in \mathbb{R}_+^\star$ (we have explicit expressions using Ergodic Theorem).
- The estimator is strongly consistent $(\alpha_T^\pm - a_\pm, \beta_T^\pm - b_\pm) \xrightarrow[T\to\infty]{\text{a.s.}} (0,0)$,
- and asymptotically normal:

$$\sqrt{T}\left(\alpha_T^\pm - a_\pm, \quad \beta_T^\pm - b_\pm\right) \xrightarrow[T\to\infty]{\text{law}} N^\pm := \left(N^{\pm,\alpha}, \quad N^{\pm,\beta}\right),$$

  where N^+, N^- are two mutually independent, independent of X, two-dimensional Gaussian r.v. with covariance matrices given by

$$2\sigma_\pm \Gamma_\pm^{-1} \text{ with } \Gamma_\pm = \begin{pmatrix} \mathcal{Q}_\infty^{\pm,-1} & -\mathcal{Q}_\infty^{\pm,0} \\ -\mathcal{Q}_\infty^{\pm,0} & \mathcal{Q}_\infty^{\pm,1} \end{pmatrix}.$$

Drift Estimation from discrete observations

We define $X_i := X_{t_i}$, with $i = 0, \cdots, N$ and set $\Delta_N = \max_{k=0\cdots,N-1}\{t_k - t_{k+1}\}$. For $m = -1, 0, 1$ and $\pm \in \{-, +\}$, let

$$\mathcal{Q}^{\pm,m}_{T,N} = \sum_{i=0}^{N-1} X_i^m 1_{\{\pm(X_i - r)\geq 0\}}(t_{k+1} - t_k),$$

$$\mathcal{M}^{\pm,m}_{T,N} = \sum_{i=0}^{N-1} X_i^m 1_{\{\pm(X_i - r)\geq 0\}}(X_{i+1} - X_i).$$

We refer with discretized likelihood to

$$\mathsf{G}_{T,N}(a_+, b_+-, a_-, b_-) = \exp\left(\sum_{i=0}^{N-1} \frac{a_r(X_i) - b_r(X_i)X_i}{2\sigma_r(X_i)X_i}(X_{i+1} - X_i) - \frac{t_{i+1} - t_i}{4}\frac{(a_r(X_i) - b_r(X_i)X_i)^2}{\sigma_r(X_i)X_i}\right)$$

We denote

$$(\alpha^{\pm}_{T,N}, \beta^{\pm}_{T,N}) = \operatorname*{Argmax}_{\theta\in\Theta} G_{T,N}(\theta).$$

Theorem MLE: Long time and high frequency

Assume $X_0 \sim \mu$, where μ is the stationary distribution. We suppose that

$$\lim_{N\to\infty} T_N = +\infty \quad \text{and} \quad \lim_{N\to\infty} \Delta_N = 0.$$

Let $\pm \in \{+,-\}$. It holds that

- The estimator is consistent $(\alpha_{T_N,N} - a_\pm, \beta_{T_N,N} - b_\pm) \xrightarrow[N\to\infty]{\mathbb{P}} (0,0)$,
- and asymptotically normal if $\lim_{N\to\infty} T_N\Delta_N = 0$:

$$\sqrt{T_N}\left(\alpha_{T_N,N} - a_\pm, \quad \beta_{T_N,N} - b_\pm\right) \xrightarrow[N\to\infty]{\text{law}} N^\pm := \left(N^{\pm,\alpha}, \quad N^{\pm,\beta}\right),$$

where N^+, N^- are two mutually independent, independent of X, two-dimensional Gaussian r.v. with covariance matrices given by

$$2\sigma_\pm\Gamma_\pm^{-1} \text{ with } \Gamma_\pm = \begin{pmatrix} \mathcal{Q}_\infty^{\pm,-1} & -\mathcal{Q}_\infty^{\pm,0} \\ -\mathcal{Q}_\infty^{\pm,0} & \mathcal{Q}_\infty^{\pm,1} \end{pmatrix}.$$

Sketch of the proof

For all $N \in \mathbb{N}$ it holds

$$\left(\alpha^{\pm}_{T_N,N} - a_{\pm}, \beta^{\pm}_{T_N,N} - b_{\pm}\right) = \left(\alpha^{\pm}_{T_N,N} - \alpha^{\pm}_{T_N}, \beta^{\pm}_{T_N,N} - \beta^{\pm}_{T_N}\right) + \underbrace{\left(\alpha^{\pm}_{T_N} - a_{\pm}, \beta^{\pm}_{T_N} - b_{\pm}\right)}_{\substack{\text{Continuous time-observations}\\ \text{Theorem}}}.$$

Making use of the Theorem on continuous time observations, we have:

$$(\alpha^{\pm}_{T_N} - a_{\pm}, \beta^{\pm}_{T_N} - b_{\pm}) \xrightarrow[N\to\infty]{\text{a.s.}} (0,0),$$

$$\sqrt{T_N}(\alpha^{\pm}_{T_N} - a_{\pm}, \quad \beta^{\pm}_{T_N} - b_{\pm}) \xrightarrow[N\to\infty]{\text{law}} N^{\pm} := \left(N^{\pm,\alpha}, \quad N^{\pm,\beta}\right),$$

We need to verify that

$$\left(\alpha^{\pm}_{T_N,N} - \alpha^{\pm}_{T_N}, \beta^{\pm}_{T_N,N} - \beta^{\pm}_{T_N}\right) \xrightarrow[N\to\infty]{\mathbb{P}} 0. \implies \text{Consistency} \tag{4}$$

$$\sqrt{T_N}\left(\alpha^{\pm}_{T_N,N} - \alpha^{\pm}_{T_N}, \beta^{\pm}_{T_N,N} - \beta^{\pm}_{T_N}\right) \xrightarrow[N\to\infty]{\mathbb{P}} 0. \implies \text{Asymptotic normality} \tag{5}$$

We can prove that (4) is verify ($\approx$ same for (5)).

Lemma (Consistency)

Let $\lim_{N\to\infty} T_N \to \infty$ *and* $\lim_{N\to\infty} \Delta_N = 0$. *Then for all* $j \in \{-1, 0\}$, $m \in \{-1, 0, 1\}$ *it holds*

$$\lim_{N\to\infty} \mathbb{E}_\mu \left[\left| \mathcal{Q}^{\pm,m}_{T_N} - \mathcal{Q}^{\pm,m}_{T_N,N} \right| \right] = 0 \text{ and } \lim_{N\to\infty} \mathbb{E}_\mu \left[\left| \mathcal{M}^{\pm,j}_{T_N} - \mathcal{M}^{\pm,j}_{T_N,N} \right| \right] = 0$$

If $X_0 = x_0 \in \mathbb{R}^+$ p.s., we need to verify:

$$\lim_{N\to\infty} \mathbb{E}_{x_0} \left[\left| \mathcal{Q}^{\pm,m}_{T_N} - \mathcal{Q}^{\pm,m}_{T_N,N} \right| \right] = 0 \text{ and } \lim_{N\to\infty} \mathbb{E}_{x_0} \left[\left| \mathcal{M}^{\pm,j}_{T_N} - \mathcal{M}^{\pm,j}_{T_N,N} \right| \right] = 0$$

$\rightarrow$ Transition probability of the TCIR remains unknown.

Conlusion

1. We prove the consistency and the asymptotic normal property for the MLE of the parameters $(a_\pm, b_\pm)$ in the positive recurrent case, and for:
    - Continuous time Observations.
    - High frequency with long time with $X_0 \sim \mu$.

Outlook

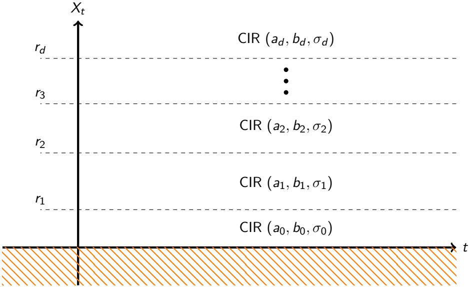
- Relax the hypothesis $X_0 \sim \mu$ for the high frequency long time estimators.
- These results can be extended to the case where $0 < a_- \leq \sigma_-$ making use of the Quasi- Likelihood [Su & Chan, 2015].
- A statistical test can be performed to verify if the process hit zero or not.
- A second statistical test can be performed to verify the existence of a threshold.
- A quadratic estimator is proposed to estimate $\sigma_\pm$ [Lejay & Pigato, 2018].

Our results can be extended to a Multi Threhsold CIR:

$$dX_t = X_0 + \int_0^t (a(X_s) - b(X_s)X_s)ds + \int_0^t \sqrt{2\sigma(X_s)X_s}dB_s, \quad t \geq 0, \quad (6)$$

with levels $-\infty = r_0 < r_1 < \cdots < r_d < r_{d+1} = +\infty$, $d \in \mathbb{N}$. Let $I_0 = (-\infty, r_1)$ and for all $j \in \{1, \cdots, d\}$, $I_j = [r_j, r_{j+1})$. The volatility and the drift coefficients are given by

$$a(x) = \sum_{j=0}^{d} a_j 1_{I_j}(x) \quad b(x) = \sum_{j=0}^{d} b_j 1_{I_j}(x) \quad \sigma(x) = \sum_{j=0}^{d} \sigma_j 1_{I_j}(x) > 0.$$



Merci pour votre attention.