Régularisation par le bruit des EDS fractionnaires

Alexandre Richard
CentraleSupélec

Avec L. Anzeletti (CS), L. Goudenège (CNRS, CS), E.M. Haress (CS) et E. Tanré (Inria, Nice)

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Overview

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Consider the equation

$$\boxed{dX_t = \varphi(X_t)dt + dB_t,}$$

when φ is a distribution in some Besov space and B is a fractional Brownian motion.

We look for solutions of the form

$$X_t = X_0 + K_t + B_t,$$

where in case φ is regular enough, $K_t = \int_0^t \varphi(X_s)ds$.

Examples

- $\varphi = \alpha\,\delta_0$: corresponds formally to an SDE involving the local time of the solution.

  In the Brownian case, SDEs involving the local time were studied by [Le Gall'84] and gives the *skew Brownian motion*:
  - when $\alpha = \pm 1$, reflected Brownian motion;
  - when $\alpha \in (-1, 1)$, process which diffuses at different speed on $\mathbb{R}_+$ and $\mathbb{R}_-$.
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- Bessel-like processes: $\varphi = \alpha\, |\cdot|^{-s}$.

Noise is your friend

Typical example

$$dX_t = \mathsf{sign}(X_t)\sqrt{|X_t|}\ \mathrm{d}t \qquad , \quad X_0 = 0,$$

whose solutions are given, for all $t^* \in \mathbb{R}_+$, by

$$(X_t^{t^*})_{t\in\mathbb{R}_+} := t \mapsto (t - t^*)_+^2.$$

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Heuristics: In situations where the ODE $\dot{x}_t = \varphi(x_t)$ lacks uniqueness, adding noise might restore uniqueness: regularisation by noise.

Rough noise is your (best) friend

For a Hurst parameter $H \in (0,1) \setminus \{\frac{1}{2}\}$, fractional Brownian motion (fBm) is given by:

$$B_t^H = c_H \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dW_s, \quad t \in \mathbb{R}.$$

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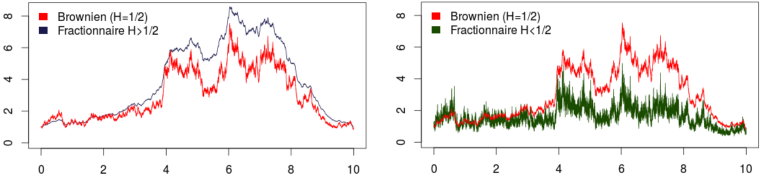
- Trajectories:



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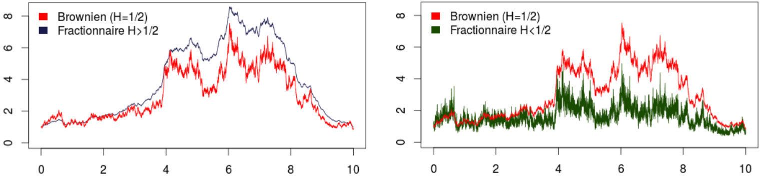

- Gaussian process with memory:
  - $H > \frac{1}{2}$: $\sum_{k=1}^{\infty} \mathbb{E}\left[B_1^H (B_{k+1}^H - B_k^H)\right] = +\infty$ (example: Niles flood).
  - Rough regime $H < \frac{1}{2}$: negatively correlated increments.

A few results

- In the Brownian case, works of Zvonkin, Veretennikov, [Krylov and Röckner'05]:

$$\text{Strong existence and uniq. for } \varphi(t,x) \in L^q_t L^p_x(\mathbb{R}^d)$$
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- In the fBm case, early work by [Nualart and Ouknine'02]. Then [Catellier and Gubinelli'16] considered *nonlinear Young differential equations* to prove that there is a unique solution if

$$\varphi \in \mathcal{C}^\gamma \text{ and } \gamma > 1 - \frac{1}{2H}.$$

  Recent extensions by Butkovsky, Galeati, Gerencser, Lê, Mytnik, etc.

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For counter-examples with $\gamma < 1 - \frac{1}{H}$ and $b \in L^{-\frac{d}{\gamma}} \hookrightarrow \mathcal{C}^{-\gamma}$, see [Butkovsky et al.'23].

Questions and objectives

- Weak well-posedness?
- Strong existence and uniqueness in case $\gamma \leq 1 - \frac{1}{2H}$?
- Numerical approximation?
- Regularity of the law of solutions?

Overview

Weak solutions

$$X_t = X_0 + \int_0^t \varphi(X_s)\,ds + B_t, \quad t \in [0,T]. \tag{1}$$

Definition

(X, B) defined on some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a weak solution of (1) if

- B is an $\mathbb{F}$-fBm;
- X is adapted to $\mathbb{F}$;
- $\exists (K_t)_{t\in[0,T]}$ such that, a.s.,

$$X_t = X_0 + K_t + B_t, \quad \forall t \in [0,T];$$

- $\forall (\varphi^n)_{n\in\mathbb{N}}$ smooth bounded functions converging to φ in $\mathcal{C}^{\gamma-}$,

$$\sup_{t\in[0,T]} \left| \int_0^t \varphi^n(X_r) dr - K_t \right| \to 0 \text{ in proba.}$$

Weak existence

Theorem ([Anzeletti, R. and Tanré'21])

Let $\varphi \in \mathcal{C}^\gamma, \gamma \in \mathbb{R}$. *Assume that*

$$\gamma > \frac{1}{2}\left(1 - \frac{1}{H}\right).$$

Then there exists a weak solution X *to* (1) *s.t.* $X - B \in \mathcal{C}^\kappa_{[0,T]}(L^m)$, $\forall \kappa \in (0, 1 + H\gamma \wedge 0] \setminus \{1\}$ *and* $m \geq 2$.

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Example

If $\varphi = \delta_0$ ($\in \mathcal{C}^{-1}$) and $d = 1$, one must choose $H < \frac{1}{3}$.
Then $X - B$ has Hölder regularity $1 - H$ ($> H$), hence X is not reflected.
See [Anzeletti'23] and [Butkovsky et al.'23] for extension to $H < \frac{1}{2}$.

The basic ingredient is the *stochastic* sewing lemma

Lemma ([Lê'20])

Let $m \in [2, \infty)$. *Let* $A : \Delta_{0,1} \to L^m(\Omega)$ *s.t.* $A_{s,t}$ *is* $\mathcal{F}_t$*-measurable.*
Assume $\exists \Gamma_1, \Gamma_2 \geq 0$, *and* $\varepsilon_1, \varepsilon_2 > 0$ *s.t.* $\forall (s,t) \in \Delta_{0,1}$ *and* $u := \frac{s+t}{2}$,

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$$\|\delta A_{s,u,t}\|_{L^m} \leq \Gamma_2 (t-s)^{\frac{1}{2}+\varepsilon_2}.$$

Then $\exists (\mathcal{A}_t)_{t \in [0,1]}$ *s.t.* $\forall t \in [0,1]$ *and any sequence of partitions* $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ *of* $[0,t]$ *with mesh size going to zero,*

$$\mathcal{A}_t = \lim_{k \to \infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in proba.}$$

Moreover, $\exists C$ *s.t.* $\forall (s,t) \in \Delta_{0,1}$,

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \leq C\,\Gamma_1 (t-s)^{1+\varepsilon_1} + C\,\Gamma_2 (t-s)^{\frac{1}{2}+\varepsilon_2},$$
$$\|\mathbb{E}[\mathcal{A}_t - \mathcal{A}_s - A_{s,t} \mid \mathcal{F}_s]\|_{L^m} \leq C\,\Gamma_1 (t-s)^{1+\varepsilon_1}.$$

Elements of proof

The SSL leads to key estimates:

$$\text{“}x \mapsto \int_0^1 f(x+B_r)\,dr \text{ is more regular than } f\text{”.}$$

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- Apply SSL to $A_{s,t} = \int_s^t f(B_r + \psi_s)\, dr$ for smooth f, (ψ_t) $\mathbb{F}$-adapted and $0 > \gamma > -1/(2H)$. Let $\alpha \in (0,1)$ s.t. $H(\gamma - 1) + \alpha > 0$.

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- For $0 > \gamma > 1 - 1/(2H)$, any $\mathcal{F}_s$-measurable $\kappa_1, \kappa_2 \in L^m(\Omega)$,
  $$\left\| \int_s^t f(B_r + \kappa_1) - f(B_r + \kappa_2) dr \right\|_{L^m} \leq C \|f\|_{\mathcal{C}^\gamma} \|\kappa_1 - \kappa_2\|_{L^m} (t-s)^{1+H(\gamma-1)},$$
  (for uniqueness).

Overview

In the Brownian case, recent results of numerical approximation for irregular drifts: e.g. [Jourdain and Menozzi'21] and [Lê and Ling'21]. See also [De Angelis et al.'19] for distributional drifts.

In the fractional case, works of Gradinaru, Neuenkirch, Nourdin, Nualart, etc. for smooth drifts. Recently, [Butkovsky et al.'21] for drifts in $\mathcal{C}^\gamma$, $\gamma > 0$.

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Let $h > 0$ and (φ^n) that approximates φ in $\mathcal{C}^{\gamma-}(\mathbb{R}^d)$. Consider the following tamed Euler scheme:

$$X_t^{h,n} = X_0 + \int_0^t \varphi^n(X_{r_h}^{h,n}) dr + B_t,$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$.

"Subcritical" case

Theorem ([Goudenège, Haress and R.'22])

Let $H < \frac{1}{2}$. *Let* $\varphi \in \mathcal{C}^\gamma$ *and assume*

$$0 > \gamma > 1 - \frac{1}{2H}.$$

Let X *denote a weak solution such that* $X - B \in \mathcal{C}^{\frac{1}{2}+H}_{[0,T]}(L^m)$, $m \geq 2$. *Let* $\varepsilon \in (0, \frac{1}{2})$. *Then* $\forall h \in (0,1)$ *and* $\forall n \in \mathbb{N}$,

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}} L^m} \leq C\Big(\|\varphi^n - \varphi\|_{\mathcal{C}^{\gamma-1}} + \|\varphi^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|\varphi^n\|_\infty \|\varphi^n\|_{\mathcal{C}^1} h^{1-\varepsilon}\Big).$$

Corollary

Let $n_h = \lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \rfloor$ *and* $\varphi^{n_h} = g_{\frac{1}{n_h}} * \varphi$*. Then* $\forall h \in (0,1)$*,*

$$[X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}} L^m} \leq C\, h^{\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon}.$$

Corollary

Let $n_h = \lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \rfloor$ *and* $\varphi^{n_h} = g_{\frac{1}{n_h}} * \varphi$*. Then* $\forall h \in (0,1)$*,*

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Example

If $\varphi = \delta_0$ ($\in \mathcal{C}^{-1}$), one must choose $H < \frac{1}{4}$. Same for any finite measure.

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Remark

Working in Besov spaces (instead of Hölder), we can get a rate for $\varphi = \delta_0$ even for $H = \frac{1}{4}$.

Rates

The orders of convergence obtained here and in [Butkovsky et al.'21] are:

Drift	*Rate*
$\gamma > 0$	$\left(\frac{1}{2} + H\gamma\right) \wedge 1 - \varepsilon$
$\gamma = 0$	$\frac{1}{2} - \varepsilon$
$\gamma \in (1 - \frac{1}{2H}, 0)$	$\frac{1}{2(1-\gamma)} - \varepsilon$
$\varphi \in \mathcal{B}_p^{\widetilde{\gamma}},\ \gamma = \widetilde{\gamma} - \frac{d}{p} = 1 - \frac{1}{2H}$ and $p < \infty$	$< H$ (non explicit)

Elements of proof (subcritical case)

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}} L^m} \leq C\Big(\|\varphi^n - \varphi\|_{\mathcal{C}^{\gamma-1}} + \|\varphi^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|\varphi^n\|_\infty \|\varphi^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \Big).$$

Denote

$$K_t^n := \int_0^t \varphi^n(X_r) dr \text{ and } K_t^{h,n} := \int_0^t \varphi^n(X_{r_h}^{h,n}) dr.$$

Decompose the error as follows:

$$\begin{aligned} X_t - X_t^{h,n} &= K_t - K_t^n && (2)\\ &\quad + \int_0^t (\varphi^n(K_r + B_r) - \varphi^n(K_r^{h,n} + B_{r_h}))dr && (3)\\ &\quad + \int_0^t (\varphi^n(K_r^{h,n} + B_{r_h}) - \varphi^n(K_{r_h}^{h,n} + B_{r_h}))dr. && (4) \end{aligned}$$

Consequence - Strong existence and uniqueness

As a consequence of the convergence of the Euler scheme:

Theorem ([Anzeletti, R. and Tanré'21], [Goudenège, Haress and R.'22])

Let $H < \frac{1}{2}$ *and* $\varphi \in \mathcal{C}^\gamma$ *(or* $\mathcal{B}_p^\gamma$*). Assume that*

$$\gamma > 1 - \frac{1}{2H} \quad \textit{(or critical condition)}.$$

- *There exists a strong solution* X*.*
- *Pathwise uniqueness holds in the class of solutions such that* $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$*.*

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- *If φ is a finite measure, pathwise uniqueness holds.*

Numerical examples in d=1

Consider

$$dX_t = \mathbb{1}_{\{X_t>0\}}dt + dB_t \qquad (5)$$

and

$$dX_t = \delta_0(X_t)dt + dB_t. \qquad (6)$$

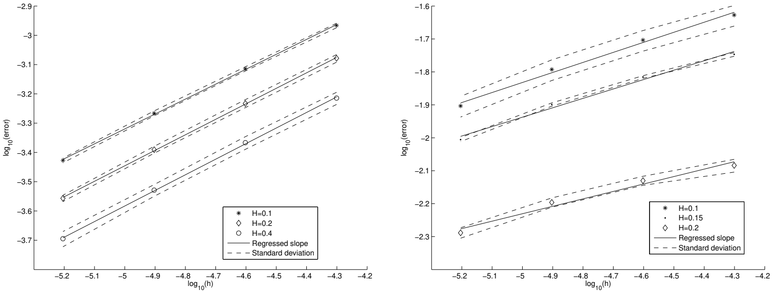

Figure: *E.M. Haress and L. Goudenège*. Log of the strong error against $\log(h)$.
Left: Equation (5). Observed slope ≈ 0.5, theoretical order $= 1/2$.
Right: Equation (6). Observed slope ≈ 0.25, theoretical order $= 1/4$.

Overview

Density of the solution

- Regularity of the density: Previous results by [Olivera-Tudor'19], [Galeati et al'22].

Theorem (Anzeletti, R. and Tanré'23+)

Let $\gamma > 1 - \frac{1}{2H}$. *Then* $\forall t > 0$, X_t *has a density and*

$$\mathcal{L}(X_\cdot) \in L^1 \mathcal{B}_1^{\frac{1}{H}-1-\varepsilon} \cap L^2 \mathcal{C}^{\frac{1}{2H}-\frac{1}{2}-d-\varepsilon}.$$

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- Towards McKean-Vlasov equations: for $b \in \mathcal{C}^{?}$,

$$\begin{cases} X_t = x + \int_0^t b * \mu_s(X_s)\, ds + B_t \\ \mu_t = \mathcal{L}(X_t). \end{cases}$$

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- Gaussian upper and lower bounds for $\gamma > 2 - \frac{1}{2H}$. See also [Li-Panloup-Sieber'23].

Stochastic heat equation with singular drift

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + \varphi(u(t,x)) + \dot{W}(t,x) \tag{7}$$

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Theorem (Goudenège, Haress and R.'23+)

The finite-differences scheme, with Euler discretisation in time for (7) *converges to* u *with a rate.*

Merci de votre attention!

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