Derivatives of killed diffusions semigroups can be represented using reflected processes

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Some background

- Kunita's flow theory: ($\partial_x f(X_T) = f'(X_T)\mathbf{D}_x X_T = f'(X_T)\mathcal{E}_T$, $b, \sigma \in C_b^1$)

$$
\begin{aligned}
X_t &= x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\
\mathcal{E}_t &= 1 + \int_0^t b'(X_s)\mathcal{E}_s ds + \int_0^t \sigma'(X_s)\mathcal{E}_s dW_s \\
\mathcal{E}_t &= \exp\left(\int_0^t \sigma'(X_s)dW_s + \int_0^t \left(b - \frac{\sigma'(X_s)^2}{2}\right) ds\right)
\end{aligned}
$$

Many examples in "regular" situations are known. (H. Kunita and others) In various cases, one is interested in computing

$$\nabla\mathbb{E}\left[f(X_T)\right] = \mathbb{E}\left[\nabla f(X_T)\mathbf{D}_x X_T\right].$$

In the present research, we are interested in one "irregular" case: The case of the elliptic killed diffusion with regular coefficients. We have divided our study in stages:

1. One dimensional case: Single barrier (Joint with D. Crisan) Local times
2. Multi-dimensional case: Half space case. No correlation at the boundary (Joint with D. Crisan) Jumps
3. Multi-dimensional case: Smooth domain. Some directions of correlation at the boundary are allowed (Joint with F. Antonelli) Covariance+Curvature

Stopped processes

Let X be a one dimensional uniformly elliptic ($\sigma(y) \geq c_0 > 0$ for all $y \in \mathbb{R}$) diffusion starting at $x > 0$. Consider $\tau := \inf\{t > 0; X_t = 0\}$.

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

Question: Is it possible to give a weak meaning to $\partial_x X_{T\wedge\tau}$?

We are not interested in distribution valued random variables.

We want a result that has an interpretation and is close to pathwise differentiation

- We need to change the notion of derivative (killing<->reflection): There exists $(Y, \mathcal{E})$ such that for every $f \in C_b^1$

$$\partial_x \mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$$

- Why change the base process?
  - The derivative of the killed process is a pair $(Y, \mathcal{E})$ such that for any $f \in C_b^1$ with $f(0) = 0$, we have

$$\partial_x \mathbb{E}\left[f(X_T) 1_{(\tau > T)}\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$$

  - In fact, it is not difficult to expect that Y has to be a reflected process. Densities of stopped (reflected) BM (no-drift !! and 1-D!!):

$$q_T^{-+}(x,y) := g_T(y-x) \underset{Y_T(refl)}{\overset{killed}{-/+}} g_T(y+x) = \left(1 \mp e^{-2\frac{yx}{T}}\right) g_T(y-x),$$

$$= (1 \pm p_T(x,y))\, g_T(y-x), \quad g_T(y) = \frac{e^{-\frac{y^2}{2T}}}{\sqrt{2\pi T}}$$

$$\partial_x \mathbb{E}\left[f(X_T) 1_{(\tau > T)}\right] = \partial_x \int_0^\infty f(y) q_T^-(x,y) dy = \int_0^\infty f'(y) q_T^+(x,y) dy = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right].$$

Strategy in a nutshell

$$\partial_x \mathbb{E}\left[f(X_T) 1_{(\tau > T)}\right] = \partial_x \mathbb{E}\left[f(X_T)(1 - p_T(x, X_T))\right] = \mathbb{E}\left[f'(X_T)(1 + p_T(x, X_T))\mathcal{E}_T\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$$

Easier said than done!

1-dim with no drift: the symmetric case

$$\begin{aligned} X_t &= x + \int_0^t \sigma(X_s) dW_s, \\ Y_t &= x + \int_0^t \sigma(Y_s) dW_s + L_t, \\ L_t &= \int_0^t 1_{(Y_s=0)} d|L|_s, \\ \mathcal{E}_t &= 1 + \int_0^t \sigma'(Y_s)\mathcal{E}_s dW_s. \end{aligned}$$

Theorem : $\partial_x \mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$

Some ideas about the proof:

- Use approximations, do the calculations and recognize the different elements that appear in the formulas.
- The technical part is to prove that second derivatives are bounded (Ascoli-Arzela)

The drift situation. (Non-symmetric)

The killed probabilistic expression remain but Harrison (1985, p. 49) states that the density of reflected Brownian motion with drift is

$$e^{-\frac{(y-x-bt)^2}{2at}} + e^{\frac{2by}{a}} e^{-\frac{(y+x+bt)^2}{2at}} - \frac{2b}{a} e^{\frac{2by}{a}} \Phi\left(\frac{y+x+bt}{\sigma\sqrt{t}}\right).$$

Here $a = \sigma^2$. What to expect?? Stopped density remains unchanged with the Girsanov weight on it. $\mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] = \tilde{\mathbb{E}}\left[f(X_T)\frac{d\mathbb{P}}{d\mathbb{Q}}1_{(\tau>T)}\right]$

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(X) = \exp\left(\int_0^T ba^{-1}(X_s)dX_s - \frac{1}{2}\int_0^T b^2a^{-1}(X_s)ds\right)$$

Change to Y gives

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(Y) = \exp\left(\int_0^T b\sigma^{-1}(Y_s)dW_s - \frac{1}{2}\int_0^T b^2a^{-1}(Y_s)ds + \int_0^T ba^{-1}(Y_s)dL_s\right)$$

Theorem

Let $f \in C_b^1$ then $\partial_x\mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] = \mathbb{E}[f'(Y_T)\mathcal{E}_T]$ where

$$\mathcal{E}_t = 1 + \int_0^t \mathcal{E}_s\left(b'(Y_s)ds + \sigma'(Y_s)dW_s + 2\frac{b}{\sigma^2}(0)\,dL_s\right).$$

The factor 2 is result of the Girsanov's theorem (+1) and the derivative of the crossing probabilities (+1).

Theorem

Let $f \in C_b^1$ *then* $\partial_x \mathbb{E}\left[f(X_T)1_{(\tau > T)}\right] = \mathbb{E}[f'(Y_T)\mathcal{E}_T]$ *where*

$$\mathcal{E}_t = 1 + \int_0^t \mathcal{E}_s \left(b'(Y_s)ds + \sigma'(Y_s)dW_s + 2\frac{b}{\sigma^2}(0)\, dL_s \right).$$

The factor 2 is result of the Girsanov's theorem (+1) and the derivative of the crossing probabilities (+1). Let $\rho_T = \sup\{s < T; Y_s = 0\} \vee 0$

$$\partial_x P_T f(x) = \partial_x \mathbb{E}_{0,x}[f(X_{T\wedge\tau})] = \mathbb{E}_{0,x}[f'(Y_T)\mathcal{E}_T] + \frac{b}{\sigma^2}(0)\, \mathbb{E}_{0,x}\left[f(Y_T)\mathcal{E}_{\rho_T} 1_{(\tau \leq T)}\right].$$

$$\partial_x P_T f(x) = \mathbb{E}[f'(Y_T)\mathcal{E}_T] + \frac{b}{a}(0)\mathbb{E}\left[\int_0^T \partial_x P_{T-s} f(0)\mathcal{E}_s dB_s\right].$$

Solve the linear equation to obtain the result

Half space case.

• In the multi-dimensional case there should be more complex boundary effects involving the normal and tangential directions. In particular, local time terms will only appear in the normal directions and in the tangential ones jump effects will also appear.

A simple example: Let us consider the domain $D = [0, \infty) \times \mathbb{R}$ and for $a, c \in \mathbb{R}$

$$X_t := (X_t^1, X_t^2) = (x_1 + aW_t^1, x_2 + cW_t^2).$$

Then using a slight extension of our results in 1D, one obtains that for $\tau(Y) := \inf\{s \geq 0; Y_s^1 = 0\}$ then for $f \in C_b^1(\mathbb{R}^2, \mathbb{R})$ with $f(0, x_2) = 0$ for all $x_2 \in \mathbb{R}$,

$$\nabla_x \mathbb{E}\left[f(X_{T\wedge\tau}) 1_{\tau^{x_1}(X) > T}\right] = \mathbb{E}\left[\nabla f(Y_T)\mathcal{E}_T\right]$$

$$\mathcal{E}_t := \begin{pmatrix} 1 & 0 \\ 0 & 1_{\tau(Y)>t} \end{pmatrix}, \ t \in [0, T].$$

This jump is a 0-th order behavior.

Lemma

The following probabilistic representation is valid for the killed process semigroup with $x_1 > L = 0$, $a_1 := \|\sigma_1\|^2$ *and* $\sigma_1 := (\sigma_{1,1}, ..., \sigma_{1,d})$.

$$\mathbb{E}\left[f(X_T)1_{(\tau > T)}\right] = \mathbb{E}\left[f(X_T)1_{(X_T^1 > 0)}\left(1 - e^{-2\frac{X_T^1 x_1}{a_1 T}}\right)\right]$$

Again the killed expression is the same. In a similar fashion, we have for the reflected process $Y_t = X_t + \ell_t \in H_L^d$

$$\begin{aligned}
\mathbb{E}\left[f(Y_T)\right] =& \mathbb{E}\left[f(X_T)1_{(X_T^1 \geq L)}\left(1 + (1 - 2\rho^2 + \xi)e^{-2\frac{X_T^1 x_1}{a_1 T}}\right)\right] \\
\rho^2 :=& \Sigma_{1,1}^{-1}\Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{2,1} \\
\xi :=& 2(1 - \rho^2)^{1/2} g\left(\frac{X_1 - x_1 - \Sigma_{1,2}\Sigma_{2,2}^{-1}(X_2 - x_2) - 2X_1(1 - \rho^2)}{(1 - \rho^2)^{1/2}\Sigma_{1,1}^{1/2} t}\right) \\
& \times \left(\frac{-2\Sigma_{1,2}\Sigma_{2,2}^{-1}(X_2 - x_2) + 2\rho^2 X_1}{\Sigma_{1,1}\sqrt{t}}\right) \\
g(x) :=& \Phi(x) e^{\frac{x^2}{2}}.
\end{aligned}$$

Hypothesis ($H_0^d = [0,\infty) \times \mathbb{R}^{d-1}$)

The functions $b, \sigma^k \in C_b^2(\mathbb{R}^d)$ are bounded, $a = (a^{ij}) = \sigma\sigma^*$ is uniformly elliptic and it satisfies $a^{1\ell}(y) = a^{\ell 1}(y) = 0$ for $y \in \partial H_0^d$ and for $\ell \neq 1$. This hypothesis implies the following condition for $\ell \neq 1$

$$a^{\ell 1}(y) = \hat{a}^{\ell}(y) y^1.$$

Here $\hat{a}^{\ell} \in C_b^1(\mathbb{R})$ and is bounded.

Theorem: $\nabla_x \mathbb{E}[f(X_T) 1_{(\tau > T)}] = \mathbb{E}[\nabla f(Y_T) \mathcal{E}_T]$.

Here, $\mathcal{E}$ solves for $\rho_t := \sup\{s < t; Y_s^1 = 0\} \vee 0$ and $\pi(y) := \sum_{\ell=2}^d \hat{a}^{\ell}(y) \mathbf{e}^{\ell} (\mathbf{e}^1)^*$

$$\begin{aligned}
\mathcal{E}_t =& I 1_{(\rho_t = 0)} + 1_{(\rho_t > 0)} \mathbf{e}^1 (\mathbf{e}^1)^* \mathcal{E}_{\rho_t -} \\
&+ \int_{\rho_t}^t \left(\mathbf{D} b(Y_s) ds + \mathbf{D}\sigma^k (Y_s) dW_s^k + 2 a^{11}(Y_s)^{-1} b^1 (Y_s) dL_s \right) \mathcal{E}_s.
\end{aligned}$$

$$\begin{aligned}
X_t =& x + \int_0^t b(X_s) ds + \int_0^t \sigma_k (X_s) dW_s^k, \\
Y_t =& x + \int_0^t b(Y_s) ds + \int_0^t \sigma_k (Y_s) dW_s^k + \mathbf{e}^1 L_t \in H_0^d,
\end{aligned}$$

BUT , if f is a function of only the first component, we have

$$\mathbb{E}\left[f(Y_T^1)\right] = \mathbb{E}\left[f(X_T^1)1_{(X_T^1>0)}\left(1 + e^{-2\frac{X_T^1 x^1}{a_1 T}}\right)\right].$$

Suppose the half space case and use the Euler scheme: $X_{i+1} = X_i + \sigma_i \Delta_i W^k$.

A way to obtain the classical result is $f_i := \mathbb{E}_i[f(X_n)]$.

$$\mathbf{D}_{i-1}\mathbb{E}_{i-1}[f_i] = \mathbb{E}_{i-1}[\mathbf{D}_i f_i(I + \mathbf{D}_{i-1}\sigma_{i-1}^k Z_i^k)].$$

Then take limits on: $\mathbf{D}_0\mathbb{E}_0[f_n] = \mathbf{D}_x\mathbb{E}[f(X_n)] = \mathbb{E}[\mathbf{D}f(X_n)\prod_{j=1}^n(I + \mathbf{D}_{j-1}\sigma_{j-1}^k Z_j^k)]$

Method of proof

For $p_i = \exp\left(-2\frac{X_i^1 X_{i-1}^1}{a_{i-1}^{11}\Delta}\right)$, consider the approximation using Girsanov

$$\mathbf{D}_x\mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] \approx \mathbf{D}_x\tilde{\mathbb{E}}\left[f(X_n)K_n\prod_{j=1}^{n}1_{(X_j^1>0)}(1-p_j)\right]$$

Iterative differentiation of $\mathbf{D}_i\tilde{\mathbb{E}}_i\left[f(X_{i+1})\kappa_{i+1}1_{(X_{i+1}^1>0)}(1-p_{i+1})\right]$gives

$$\mathbf{D}_x\tilde{\mathbb{E}}\left[f(X_n)K_n\prod_{j=1}^{n}1_{(X_j^1>0)}(1-p_j)\right] \approx \tilde{\mathbb{E}}\left[\mathbf{D}f(X_n)E_nK_n\prod_{j=1}^{n}1_{(X_j^1>0)}(1+p_j)\right]$$
$$+\sum_{i=1}^{n}\tilde{\mathbb{E}}\left[f(X_n)\frac{b_{i-1}}{a_{i-1}}1_{(X_i^1>0)}p_iE_{i-1}K_n\prod_{j=1}^{i-1}1_{(X_j^1>0)}(1+p_j)\prod_{j=i+1}^{n}1_{(X_j^1>0)}(1-p_j)\right]$$

Take limits. The important point appears in the constant terms of $E_n = \prod_{j=1}^n e_j$

$$e_i := I(1-p_i) + \mathbf{e}^1(\mathbf{e}^1)^\top p_i + \bar{e}_i$$
$$\bar{e}_i := r_i + \mathbf{D}b_{i-1}\Delta\,(1-p_i) + \pi_{i-1}X_{i-1}^1 p_i$$
$$r_i := (1-p_i)\,\mathbf{D}\sigma_{i-1}^k\Delta_i W^k + X_{i-1}^1\sigma_{i-1}^k\mathbf{D}_{i-1}\left(\frac{\sigma_{i-1}^{1k}}{a_{i-1}^{11}}\right)p_i$$

DIFFERENTIATION OF CROSSING PROBABILITIES

Let $G_i := g(X_{i-1}, \Delta_i W^k)$ for $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a Lipschitz function. Then first, we have the following representation for $\tau(i-1) := \inf\{s > t_{i-1}; X_s^1 \leq 0\}$ where $X_i = X_{i-1} + \sigma_{i-1}^k W_i^k$. In this case $p_i = \exp\left(-2\frac{X_i^1 X_{i-1}^1}{a_{i-1}^{11}\Delta}\right)$ is the crossing probability for half space $\mathcal{D} = [0, \infty) \times \mathbb{R}^{d-1}$. That is,

$$\mathbb{E}_{i-1}\left[G_i 1_{(\tau(i-1)>t_i)}\right] = \mathbb{E}_{i-1}\left[G_i(1-p_i)\right]$$

Differentiation gives:(D_i^k denotes derivative wrt to the k-th noise)

$$\mathbb{E}_{i-1}\left[G_i \mathbf{D}_{i-1} p_i\right] = -2\mathbb{E}_{i-1}\left[D_i^k G_i \mathbf{D}_{i-1}\left(\frac{\sigma_{i-1}^{1k} \mathtt{X}_{i-1}^1}{a_{i-1}^{11}}\right) p_i\right],$$

In the above situation $G_i = f_i \kappa_i$.

Therefore the above terms will also create local time and jumps .

local time: $\mathbf{D}_{i-1}\left(\frac{\sigma_{i-1}^{1k}}{a_{i-1}^{11}}\right) X_{i-1}^1 p_i$

jumps + local time: $\frac{\sigma_{i-1}^{1k} \mathbf{D}_{i-1} X_{i-1}^1}{a_{i-1}^{11}} p_i$

Girsanov term e^{κ_i}, introduces more local time terms.

Conclusions and Comments

- Derivatives of killed processes can jump everytime there is a chance that the process may touch the boundary. The directions of jump are determined by the interaction between correlation and the boundary geometry.
- Derivatives are affected by local time due to interaction of the boundary with drift and correlation direction.
- In this sense, curvature of the boundary plays an important role in the local time terms.
- The technically difficult part is to prove boundedness of second derivatives for the approximating sequence and recognizing each term in the method of proof.
- From the method of proof one can also obtain BEL and IBP formulas based on excursions. In particular, the last excursion from the boundary. But note that this is only the elliptic case. For the hypoelliptic case, compare with Thalmaier and coauthors.
- "Dual" descriptions are available for derivatives of reflected processes
- Future projects: Description using pde's will be difficult in gemeral due to the jumps and the non-adapted ρ_t (honest time). Uniqueness? Exit times?

References I

[1] Andres, Sebastian. Pathwise Differentiability for SDEs in a Smooth Domain with Reflection. Electron. J. Probab. 16 (2011), paper no. 28, 845–879.

[2] Airault, Hélène and Malliavin, Paul and Ren, Jiagang, J. Math. Pures Appl., 1069–1091, Smoothness of stopping times of diffusion processes, 78, 1999

[3] Bismut, Jean -Michel, Last exit decompositions and regularity at the boundary of transition probabilities, Z. Wahrscheinlichkeitstheorie verw Gebiete, 1985, 69, 65–98.

[4] Cattiaux, Patrick, Ann. Inst. Fourier (Grenoble), 541–624, Stochastic calculus and degenerate boundary value problems, 42, 1992

[5] Cattiaux, Patrick, Hypoellipticité et hypoellipticité partielle pour les diffusions avec une condition frontière, Ann. Inst. H. Poincaré Probab. Statist., 22,1986,67–112

[6] JD Deuschel, L Zambotti. Bismut–Elworthy's formula and random walk representation for SDEs with reflection. Stochastic processes and their applications 115 (6), 907-925

[7] Elworthy, K. D. and Li, X.-M. Formulae for the derivatives of heat semigroups. J. Funct. Anal., 125, 1994, 252–286,

[8] N. Frikha, A. Kohatsu-Higa, L. Li. Integration by parts formula for killed processes: a point of view of approximation theory. Electronical Journal of Probability. Vol. 24, paper no. 95, 1-44, 2019

[9] Emmanuel Gobet. Euler schemes and half-space approximation for the simulation of diffusion in a domain. ESAIM: PS Volume 5, 2001, 261 - 297

[10] Jacod, Jean. Rates of convergence to the local time of a diffusion. Ann. Inst. H. Poincaré Probab. Statist., 34 (1998) no. 4, pp. 505-544

[11] Thomas G. Kurtz, Philip Protter .Weak Limit Theorems for Stochastic Integrals and Stochastic Differential Equations, Ann. Probab. 19(3), 1035-1070, (1991)

[12] Hiroshi Kunita. Stochastic Flows and Stochastic Differential Equations. Cambridge Studies in Advanced Mathematics, 1997

[13] Lépingle, D. Euler scheme for reflected stochastic differential equations. Math. Comput. Simulation 38 (1995) 119-126.

[14] Lipshutz, D. and Ramanan, K. Pathwise differentiability of reflected diffusions in convex polyhedral domains. Ann. Inst. H. Poincaré Probab. Statist. 55(3): 1439-1476 (2019).

[15] Malliavin, P; Thalmaier, A. Stochastic calculus of variations in mathematical finance. Springer (2006)

[16] A. Jakubowski, J. Mémin and G. Pagès, Convergence en loi des suites d'intégrales stochastiques sur l'espace ID1 de Skorokhod, Probab. Theory Related Fields, 81, p. 111-137, 1989.

[17] Pilipenko, A. Stochastic flows with reflection. (Russian) Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 2005, no. 10, 23-28.

[18] Pilipenko, A. Properties of flows generated by stochastic equations with reflection. (Russian) Ukraïn. Mat. Zh. 57 (2005), no. 8, 1069-1078; translation in Ukrainian Math. J. 57 (2005), no. 8, 1262-1274