Genealogy of a random walk on a Galton-Watson tree in random environment

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Journées de Probabilités 2023

19 juin 2023

Outline

Introduction

A Galton-Watson marked tree $(\mathbb{T}, (M(u); u \in \mathbb{T})$

Under a probability P, let (N, M) be a random variable taking values in $\mathbb{N} \times (0, \infty)$, the offspring N satisfies $\mathsf{E}[N] > 1$.

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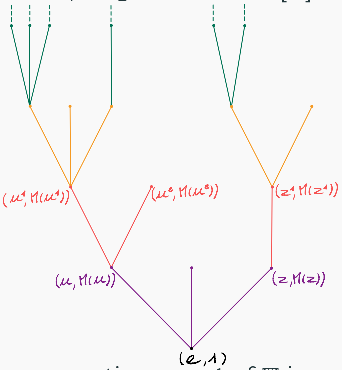

Let $m \in \mathbb{N}^*$. If the generation $m - 1$ of $\mathbb{T}$ is not empty, then any vertex $u \in \mathbb{T}$ in the generation $n - 1$ gives progeny to N_u marked children $(u^1, M(u^1)), \ldots, (u^{N_u}, M(u^{N_u}))$ independently of other vertices in generation $n - 1$ according to the law of (N, M), thus forming the generation m.

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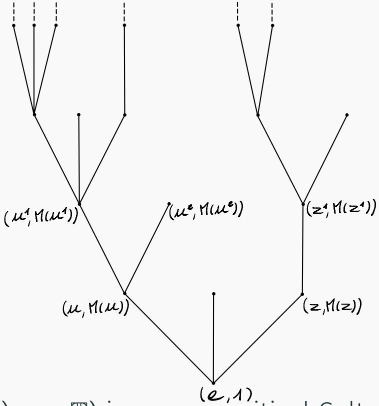
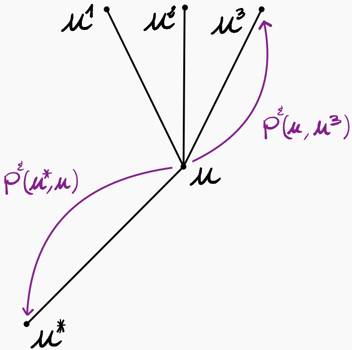


$\mathcal{E} = (\mathbb{T}, (M(u); u \in \mathbb{T})$ is a super-critical Galton-Watson marked tree and let $\mathsf{P}^* := \mathsf{P}(\cdot|$ non-extinction $)$.

Random walk on $\mathcal{E}$

For a realization of $\mathcal{E} = (\mathbb{T}, (M(u); u \in \mathbb{T}))$, introduce a $\mathbb{T} \cup \{e^*\}$-valued random walk $X = (X_j)_{j \in \mathbb{N}}$ under the quenched probability $\mathbb{P}^{\mathcal{E}}$, starting from e and reflected in e^* with the following transition probabilities: for any $u \neq e^*$

$$p^{\mathcal{E}}(u, u^*) = \frac{1}{1 + \sum_{j=1}^{N_u} M(u^j)}, \; p^{\mathcal{E}}(u, u^i) = \frac{M(u^i)}{1 + \sum_{j=1}^{N_u} M(u^j)}.$$



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Define the annealed probabilities

$$\mathbb{P} := \mathsf{E}[\mathbb{P}^{\mathcal{E}}(\cdot)] \quad \text{and} \quad \mathbb{P}^* := \mathsf{E}^*[\mathbb{P}^{\mathcal{E}}(\cdot)].$$

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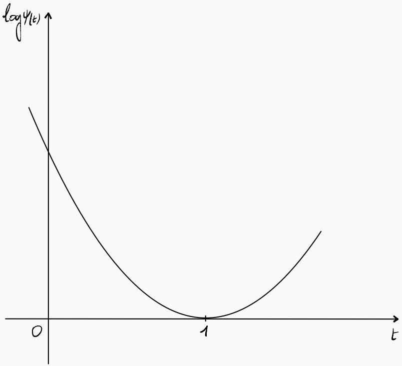

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$$\frac{1}{(\log n)^3} \max_{j \le n} |X_j| \underset{n\to\infty}{\longrightarrow} \frac{8}{3\pi^2 \psi''(1)} \text{ (Faraud, Hu, Shi 2012)},$$

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and $(|X_n|/(\log n)^2)_{n\geq 2}$ converges in law under $\mathbb{P}^*$ to a positive random variable (Hu, Shi 2016).

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$(\frac{\log n}{n}\mathcal{L}^n)_{n\in\mathbb{N}^*}$ converges in $\mathbb{P}^*$-probability (Hu, Shi 2016).

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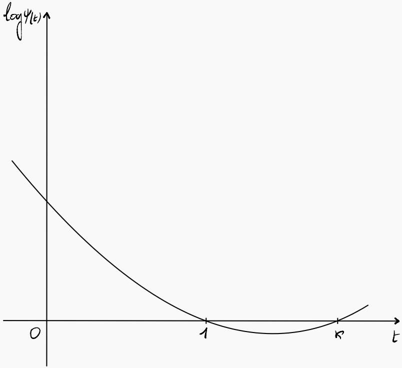

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$\left(|X_n|/n^{1-\frac{1}{\kappa\wedge 2}}\right)_{n\in\mathbb{N}^*}$ if $\kappa \neq 2$ and $\left(\left(\frac{\log n}{n}\right)^{1/2}|X_n|\right)_{n\in\mathbb{N}^*}$ if $\kappa = 2$ converge in law under $\mathbb{P}^*$ (Faraud 2011; Aïdékon, de Raphélis 2017 and de Raphélis 2022).

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$(\mathcal{L}^n / n^{\frac{1}{\kappa \wedge 2}})$ if $\kappa \neq 2$ and $(\mathcal{L}^n / (n \log n)^{1/2})$ if $\kappa = 2$ converge in law under $\mathbb{P}^*$ (Hu 2017; K 2023+).

Range of the diffusive random walk on $\mathcal{E}$

Let $T \in \mathbb{N}^*$ and introduce the sub-tree $\mathcal{R}_T := \{u \in \mathbb{T}; \exists j \leq T : X_j = u\}$ of $\mathbb{T}$. $\mathcal{R}_T$ is the range of X up to the time T.

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$$\frac{1}{n}\#\mathcal{R}_n \underset{n\to\infty}{\longrightarrow} \text{constant } C_1 > 0,$$

and, in law

$$\frac{C_2}{n^{1/2}}\mathcal{R}_n \underset{n\to\infty}{\longrightarrow} \mathcal{T}_{(|B_t|)_{t\in[0,1]}},$$

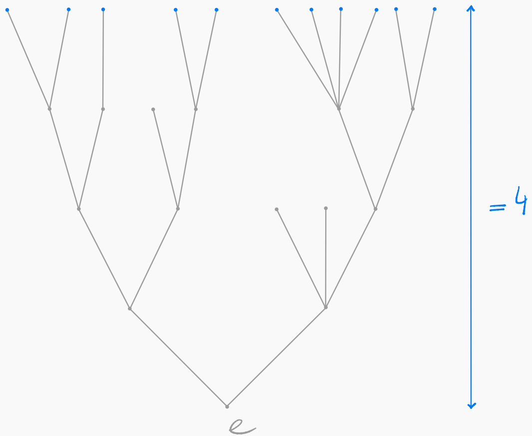
where $C_1, C_2 > 0$ are explicit constants.

Generalized range in the diffusive case

Constraints on visited vertices

Let $m \in \mathbb{N}^*$. Note $\mathbb{T}_m$ the m-th generation of the $\mathbb{T}$:

$$\mathbb{T}_m = \{u \in \mathbb{T} : \ |u| = m\}.$$



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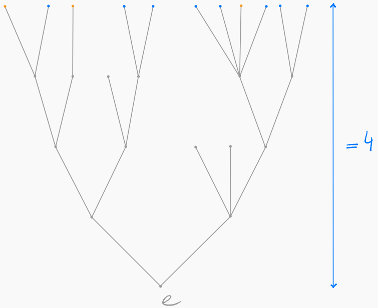
For any $k \in \mathbb{N}$, $k \geq 2$, introduce

$$\Delta_m^k := \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in (\mathbb{T}_m)^{\times k};\ \forall\, i \neq j,\ x^{(i)} \neq x^{(j)}\},$$

the set of k-tuples of distinct vertices of $\mathbb{T}_m$.



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Let $n \in \mathbb{N}$ and $\mathbb{B}_n^k \subset \mathbb{T}^{\times k}$. Introduce the following range

$$R_n^k := \sum_{\mathbf{x} \in \Delta_{m_n}^k} \mathbb{1}_{\{\mathcal{L}_{\mathbf{x}}^n \geq 1\}} \mathbb{1}_{\{\mathbf{x} \in \mathbb{B}_n^k\}},$$

where, for any $\mathbf{x} = (x^{(1)}, \ldots, x^{(k)}) \in \Delta_m^k$, $\mathcal{L}_{\mathbf{x}}^n = \min_{1 \leq i \leq k} \mathcal{L}_{x^{(i)}}^{\tau^n}$.

Constraints on visited vertices

Let $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(k)}) \in \Delta_m^k$. Define the first coalescent time $\mathcal{S}^k(\boldsymbol{x}) - 1$ of the vertices $x^{(1)}, \ldots, x^{(k)}$ by

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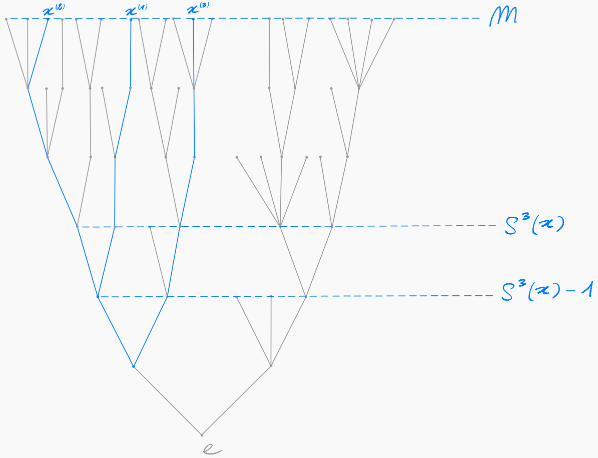

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Let $p \in \{1, \ldots, m\}$. Introduce the subset $\mathcal{C}_{m,p}^k$ of Δ_m^k

$$\mathcal{C}_{m,p}^k := \{\boldsymbol{x} \in \Delta_m^k;\ \mathcal{S}^k(\boldsymbol{x}) \leq p\}.$$

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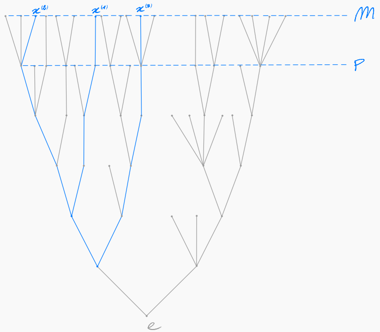

Constraints on visited vertices

We assume here that for all $n \in \mathbb{N}^*$, $\mathbb{B}_n^k = \mathbb{B}^k$ where the set $\mathbb{B}^k$ satisfies the following hereditary hypothesis:

Hereditary hypothesis

There exists $\mathfrak{g} \in \mathbb{N}^*$ such that for any $p \geq \mathfrak{g}$, if $|x^{(i)}| = m \geq p$ and $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \mathcal{C}_{m,p}^k$, then

$$\boldsymbol{x} \in \mathbb{B}^k \iff \boldsymbol{x}_p \in \mathbb{B}^k,$$

where $\boldsymbol{x}_p = ((x^{(1)})_p, \dots, (x^{(k)})_p)$.

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There exists $\mathfrak{g} \in \mathbb{N}^*$ such that for any $p \geq \mathfrak{g}$, if $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(k)}) \in \mathcal{C}^k_{m,p}$ (with $m \geq p$), then

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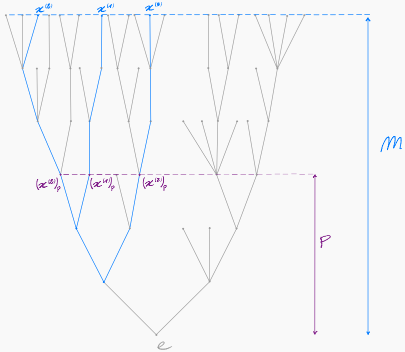

Constraints on visited vertices: a general result

Theorem (K 23)

Let $m_n = o(n^{1/2})$, $m_n \geq \delta \log n$ for some $\delta > 0$. Assume $\kappa > 2k$. If the hereditary hypothesis holds, then, in $\mathbb{P}^*$-probability

$$\frac{1}{n^{k/2}} \sum_{x \in \mathbf{\Delta}^k_{m_n}} \mathbb{1}_{\{\mathcal{L}^n_x \geq 1\}} \mathbb{1}_{\{x \in \mathbb{B}^k\}} \underset{n \to \infty}{\longrightarrow} (c_\infty)^k \mathcal{A}_\infty(\mathbb{B}^k),$$

where $c_\infty > 0$ is a constant and

$$\mathcal{A}_\infty(\mathbb{B}^k) = \lim_{\ell \to \infty} \sum_{x \in \mathbf{\Delta}^k_\ell} \mathbb{1}_{\{x \in \mathbb{B}^k\}} \prod_{i=1}^{k} \prod_{e < z \leq x^{(i)}} A(z).$$

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$$\frac{1}{n^{k/2}} \sum_{x\in\mathcal{C}^k_{m_n,p}} \mathbb{1}_{\{\mathcal{L}^n_x \geq 1\}} \underset{n\to\infty}{\longrightarrow} (c_\infty)^k \mathcal{A}_\infty(\{x;\ \mathcal{S}^k(x) \leq p\}) \quad \text{in } \mathbb{P}^*\text{-probability},$$

  and $\lim_{p\to\infty} \mathcal{A}_\infty(\{x;\ \mathcal{S}^k(x) \leq p\}) = (c_\infty W_\infty)^k$.

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- $k \geq 3$, $\lambda_2, \ldots, \lambda_k \in \mathbb{N}^*$ and $\mathbb{B}^k = \bigcap_{i=2}^k \{\mathbf{x};\ |x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}$ etc.

The critical generations

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Proposition (K 23+)

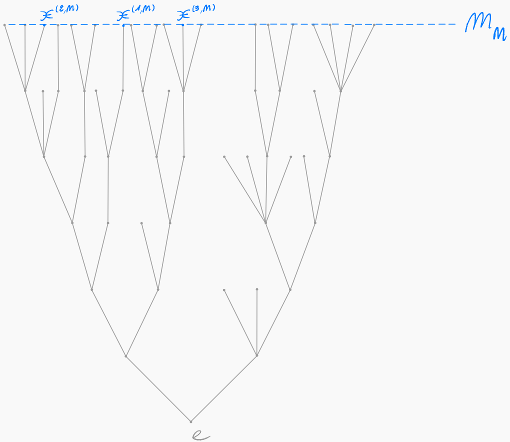
In law, under $\mathbb{P}^*$

$$\frac{1}{n^{1/2}} \#(\mathbb{T}_{m_n} \cap \mathcal{R}_{\tau^n}) \underset{n\to\infty}{\longrightarrow} c_\infty W_\infty Z_\alpha,$$

where $(Z_t)_{t\geq 0}$ is a CSBP starting from 1 with branching mechanism $\lambda \mapsto \frac{c_0}{W_\infty}\lambda^2$ for some constant $c_0 > 0$.

Application: a genealogy problem

Pick $k \geq 2$ vertices $\mathfrak{X}^{(1,n)}, \ldots, \mathfrak{X}^{(k,n)}$ uniformly and without replacement in $\{u \in \mathcal{R}_{\tau^n};\ |u| = m_n\}$.



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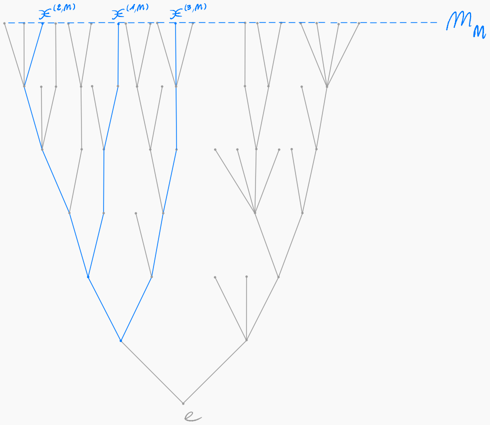

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$$\lim_{n \to \infty} \mathbb{P}^*\left(\mathfrak{X}^{(n)} \in \mathbb{B}^k\right) = \mathsf{E}^*\left[\frac{\mathcal{A}_\infty(\mathbb{B}^k)}{(W_\infty)^k}\right],$$

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In particular, for any $p \in \mathbb{N}^*$

$$\lim_{n\to\infty} \mathbb{P}^*\big(\mathcal{S}^k(\mathfrak{X}^{(n)}) \leq p\big) = \mathsf{E}^*\Big[\frac{\mathcal{A}_\infty(\{x;\ \mathcal{S}^k(x) \leq p\})}{(W_\infty)^k}\Big],$$

with $\lim_{p\to\infty} \mathsf{E}^*\big[\frac{\mathcal{A}_\infty(\{x;\ \mathcal{S}^k(x)\leq p\})}{(W_\infty)^k}\big] = 1.$

Application: a genealogy problem

Pick $k \geq 2$ vertices $\mathfrak{X}^{(1,n)}, \dots, \mathfrak{X}^{(k,n)}$ uniformly and without replacement in $\{u \in \mathcal{R}_{\tau^n};\ |u| = m_n\}$.

What does their genealogical tree look like? Define the random k-tuples $\mathfrak{X}^{(n)} := (\mathfrak{X}^{(1,n)}, \dots, \mathfrak{X}^{(k,n)})$.

In particular, for any $p \in \mathbb{N}^*$

$$\lim_{n\to\infty} \mathbb{P}^*\big(\mathcal{S}^k(\mathfrak{X}^{(n)}) \leq p\big) = \mathsf{E}^*\Big[\frac{\mathcal{A}_\infty(\{\mathsf{x};\ \mathcal{S}^k(\mathsf{x}) \leq p\})}{(W_\infty)^k}\Big],$$

with $\lim_{p\to\infty} \mathsf{E}^*\big[\frac{\mathcal{A}_\infty(\{\mathsf{x};\ \mathcal{S}^k(\mathsf{x}) \leq p\})}{(W_\infty)^k}\big] = 1.$

Similar behaviour as the one observed for the genealogy of a regular Gaton-Watson super-critical tree (Athreya 2012; Johnston 2019; Harris, Johnston, Roberts 2020).

Application: a genealogy problem

Pick 2 vertices $\mathfrak{X}^{(1,n)}$ and $\mathfrak{X}^{(2,n)}$ uniformly and without replacement in $\{u \in \mathcal{R}_{\tau^n};\ |u| = m_n\}$.

Theorem (Andreoletti, K 23+)

Let $m_n = n^{1/2}$, $\ell \in \mathbb{N}^*$ and $b \in (0,1)$. The two following limits exist:

$$p_\ell := \lim_{n\to\infty} \mathbb{P}^*\left(|\mathfrak{X}^{(1,n)} \wedge \mathfrak{X}^{(2,n)}| < \ell\right)$$

and

$$q_b := \lim_{n\to\infty} \mathbb{P}^*\left(|\mathfrak{X}^{(1,n)} \wedge \mathfrak{X}^{(2,n)}| \geq b n^{1/2}\right).$$

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Moreover

- $\lim_{\ell\to\infty} p_\ell \in (0,1)$, $\lim_{b\to 0} p_b \in (0,1)$;
- $\lim_{\ell\to\infty} p_\ell + \lim_{b\to 0} p_b = 1$.

Merci!