Spectral gap for log-concave measures on convex domains

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General setting

- Consider a compact domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ on which is defined a probability measure

$$d\mu(x) \propto e^{-V(x)} dx,$$

where V is some smooth potential on Ω.

$\hookrightarrow$ Canonical diffusion operator, endowed with Neumann boundary condition: if $\eta(x)$ is the outer unit normal vector at point $x \in \partial\Omega$,

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle, \quad \langle \nabla f, \eta \rangle = 0 \text{ on } \partial\Omega,$$

which is (essentially) self-adjoint in $L^2(\mu)$:

$$\int_\Omega f \, Lg \, d\mu = \int_\Omega Lf \, g \, d\mu = - \int_\Omega \langle \nabla f, \nabla g \rangle \, d\mu.$$

- Spectrum : $\sigma(-L) \subset [0, \infty)$ with $\lambda_0(-L) = 0$ (associated to const).

- Spectral gap: given $\lambda > 0$, we have

$$\sigma(-L) \subset \{0\} \cup [\lambda, \infty),$$

iif the Poincaré inequality holds with constant λ: for all $f \perp$ const,

$$\lambda \int_\Omega f^2 \, d\mu \quad \leq \quad \int_\Omega |\nabla f|^2 \, d\mu.$$

- Optimal constant $\lambda_1(\Omega, \mu)$, called the spectral gap (of $-L$).
- Describes the speed of convergence to equilibrium in $L^2(\mu)$ of the law of the underlying Markov process solution to the SDE

$$dX_t = -\nabla V(X_t) \, dt + \sqrt{2} \, dB_t,$$

reflected at the boundary, where $(B_t)_{t \geq 0}$ is a standard Brownian motion on $\mathbb{R}^n$.

Examples

- The uniform distribution on a convex body Ω (*i.e.*, a compact convex set with non-empty interior):

$$V \equiv 0 \quad \text{and} \quad Lf = \Delta f.$$

- Log-concave probability measures (V is convex) truncated on a domain Ω.
  - The standard Gaussian case:

$$V(x) = \frac{|x|^2}{2} \quad \text{and} \quad Lf = \Delta f - \langle x, \nabla f \rangle.$$

  - The Subbotin distribution of parameter $\alpha \in [1, \infty)$:

$$V(x) = \frac{|x|^\alpha}{\alpha} \quad \text{and} \quad Lf = \Delta f - |x|^{\alpha-2} \langle x, \nabla f \rangle.$$

A classical result relying on convexity

- An instance of Bakry-Émery curvature-dimension criterion (Bakry-Émery '85): if Ω is convex and V is uniformly convex, *i.e.*,

$$\mathrm{Hess}\, V \geq \lambda\, I_n,$$

for some $\lambda > 0$, then

$$\lambda_1(\Omega, \mu) \geq \lambda.$$

- In the standard Gaussian case, we have on $\mathbb{R}^n$,

$$\lambda_1(\Omega, \mu) = 1,$$

whereas on any convex Ω,

$$\lambda_1(\Omega, \mu) \geq 1.$$

Why is the convexity of Ω so important ?

$\hookrightarrow$ It avoids bottlenecks ! (without convexity, the time to reach equilibrium for the process could be very long).

In many situations, there exists some smooth function F on $\mathbb{R}^n$ such that

$$\Omega = \{F \leq 0\} \quad \text{and} \quad \partial\Omega = \{F = 0\}.$$

Then the outer unit normal vector $\eta(x)$ at point $x \in \partial\Omega$ is given by

$$\eta(x) = \frac{\nabla F(x)}{|\nabla F(x)|}.$$

The second fundamental form (curvature) $\mathrm{Jac}\,\eta(x)$ acts on the tangent space $\eta(x)^\perp$ as

$$\mathrm{Jac}\,\eta(x)|_{\eta(x)^\perp} \quad = \quad \frac{\mathrm{Hess}\,F(x)}{|\nabla F(x)|},$$

which is ≥ 0 as soon as F is convex, meaning that Ω is a convex set.

Question: How to obtain a spectral gap result beyond the uniform convexity of V ?

- A huge literature on the subject since almost 40 years using modern tools of analysis, geometry and probability.
- Pioneers: H. Poincaré at the end of the 19th century, Payne-Weinberger '60, Gromov-Milman '87.
- Kannan-Lovász-Simonovits '95, leading then to the famous KLS conjecture (still open, but under progress).

  In general, we don't really care about dimension-free constants.
  $\hookrightarrow$ **Not suitable for the Global Sensitivity Analysis methodology in Statistics.**

Main difficulties in high dimension:

- The curse of dimensionality.
- To handle the geometry of Ω.
- No monotonicity property with respect to the inclusion of domains.

What is known for the uniform distribution on a compact domain ?

$\hookrightarrow$ Only few explicit results:

- Hypercubes (and also hyperrectangles): by tensorization,

$$\lambda_1([-R,R]^n) = \lambda_1([-R,R]) = \frac{\pi^2}{4R^2}.$$

- Euclidean balls:

$$\lambda_1(\mathcal{B}(0,R)) = \frac{p_{n/2}^2}{R^2} \asymp \frac{n}{R^2}.$$

- Some two-dimensional triangles.

What is known for the uniform distribution on a compact domain ?

$\hookrightarrow$ Few totally explicit bounds, only for convex bodies:

- Payne-Weinberger '60:

$$\lambda_1(\Omega) \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}.$$

- Klartag '09 for unconditional monotonicity: if $\Omega \subset [-R, R]^n$ is unconditional then

$$\lambda_1(\Omega) \geq \lambda_1([-R, R]^n).$$

- Various stability results by E. Milman '09.

Comparison result for compact domains $\Omega \subset \mathbb{R}^n$, cf. Weinberger '56:

$$\lambda_1(\Omega) \leq \left(\frac{\mathrm{vol}(\mathcal{B}(0,1))}{\mathrm{vol}(\Omega)} \right)^{2/n} \lambda_1(\mathcal{B}(0,1)),$$

i.e., the spectral gap of Euclidean balls maximizes the spectral gap of compact domains Ω with the same volume.

Question: Given $\Omega \subset \mathbb{R}^n$ such that $\mathrm{vol}(\Omega) = \mathrm{vol}(\mathcal{B}(0,1))$, which assumptions on Ω ensure

$$\lambda_1(\Omega) \asymp \lambda_1(\mathcal{B}(0,1))\,?$$

Our answer: Uniformly convex bodies, *i.e.*, convex bodies such that the second fundamental form is uniformly bounded from below: there exists $\rho > 0$ such that

$$\mathrm{Jac}\,\eta(x)_{|_{\eta(x)^\perp}} \geq \rho\, I_n, \quad x \in \partial\Omega.$$

$\hookrightarrow$ Example: the ball $\mathcal{B}(0,R)$ is uniformly convex with $\rho = 1/R$.

Theorem (Bonnefont-J. '22)

*Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex body with the origin in its interior. There exists some **explicit** constant $C \in (0,1)$, depending only on ρ and on the position of the origin, such that the inequality holds*

$$\lambda_1(\Omega) \quad \geq \quad C \left(\frac{\mathrm{vol}(\mathcal{B}(0,1))}{\mathrm{vol}(\Omega)} \right)^{2/n} \lambda_1(\mathcal{B}(0,1)).$$

Recall that the Subbotin distribution μ_α of parameter $\alpha \in [1, \infty)$ has Lebesgue density proportional to e^{-V} on Ω, where

$$V(x) = \frac{|x|^\alpha}{\alpha} \quad \text{and} \quad Lf = \Delta f - |x|^{\alpha-2} \langle x, \nabla f \rangle.$$

Theorem (Bonnefont-J. '22)

$$\lambda_1(\mathcal{B}(0,R), \mu_\alpha) \asymp \max\left\{\frac{n}{R^2}, n^{1-\frac{2}{\alpha}}\right\},$$

the transition being at $R \approx n^{1/\alpha}$ *(expected Euclidean norm on* $\mathbb{R}^n$*). In other words,*

$$\lambda_1(\mathcal{B}(0,R), \mu_\alpha) \asymp \max\left\{\lambda_1(\mathcal{B}(0,R)), \lambda_1(\mathbb{R}^n, \mu_\alpha)\right\}.$$

Beyond the convex case in the Gaussian setting.

Theorem (Bonnefont-J. '22)

$$\lambda_1(\mathbb{R}^n \setminus \mathcal{B}(0,R), \mu_2) \asymp \min\left\{\frac{n}{R^2}, 1\right\},$$

i.e.,

$$\lambda_1(\mathbb{R}^n \setminus \mathcal{B}(0,R), \mu_2) \asymp \min\left\{\lambda_1(\mathcal{B}(0,R)), \lambda_1(\mathbb{R}^n, \mu_2)\right\}.$$

What about the proofs ?

- Intertwining approach (developed with M. Bonnefont during the last years).
- Dualization of the Poincaré inequality.

Intertwining in the 1d case, without boundary:

From a probabilistic point of view:

$$\begin{aligned} \partial_x \mathbb{E}\left[f(X_t^x)\right] &= \mathbb{E}\left[f'(X_t^x)\,\partial_x X_t^x\right] \\ &= \mathbb{E}\left[f'(X_t^x)\, e^{-\int_0^t V''(X_s^x)\,ds}\right] \quad \text{(tangent process method)} \\ &= \mathbb{E}\left[f'(X_{a,t}^x)\, e^{-\int_0^t V''(X_{a,s}^x)\,ds}\, M_{a,t}\right] \quad \text{(Doob's transform)} \end{aligned}$$

where $(X_{a,t}^x)_{t\geq 0}$ is the diffusion process with generator

$$L_a f = f'' - \left(V + \log(a^2)\right)'\, f',$$

and $(M_{a,t})_{t\geq 0}$ is the Girsanov martingale

$$M_{a,t} = \frac{a(X_{a,t}^x)}{a(x)}\, e^{-\int_0^t \frac{L_a(a)}{a}(X_{a,s}^x)\,ds} = \frac{a(X_{a,t}^x)}{a(x)}\, e^{+\int_0^t a\,L(1/a)\,(X_{a,s}^x)\,ds}.$$

Intertwining in the 1d case, without boundary:

Hence the intertwining with weight a reads as the following identity: if we denote the weighted gradient $\partial_a = a\,\partial_x$,

$$\partial_a \mathbb{E}\left[f(X_t^x)\right] = \mathbb{E}\left[\partial_a f(X_{a,t}^x)\, e^{-\int_0^t (V'' - a\,L(1/a))(X_{a,s}^x)\,ds}\right].$$

Finally choosing $a = 1/g'$ (*i.e.*, $\partial_a g = 1$) for some smooth function g with non-vanishing derivative, then

$$V'' - a\,L(1/a) = -\frac{(Lg)'}{g'},$$

and taking g as the eigenfunction (if it exists !) associated to the spectral gap yields

$$V'' - a\,L(1/a) \equiv \lambda_1(\mathbb{R}, \mu).$$

As predicted by Jim Morrison, this is the end...

THANK YOU
FOR YOUR ATTENTION