Patterns crossed by geodesics in first-passage percolation

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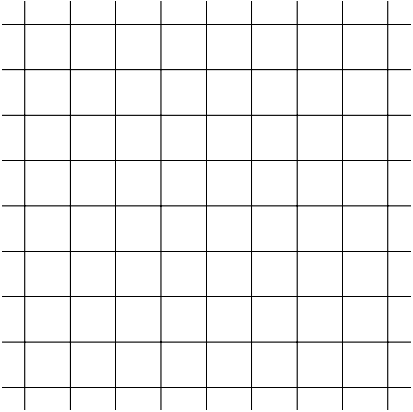
Journées de Probabilités, 20 juin 2023

The model of first passage percolation

Hammersley and Welsh, 1965.



Lattice $\mathbb{Z}^d$, $d \geq 2$.

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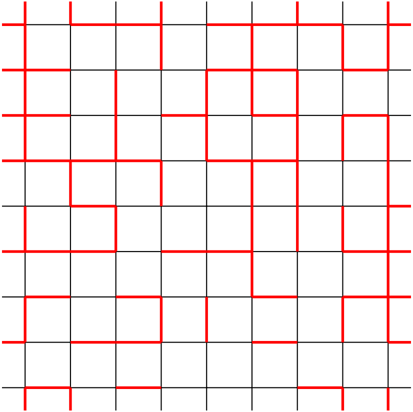

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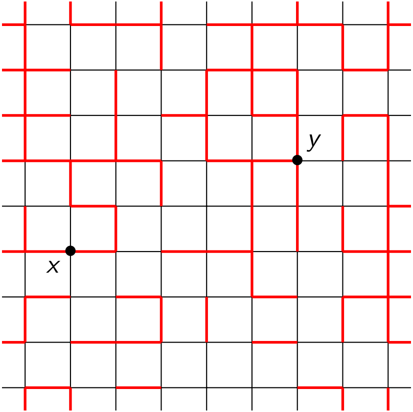

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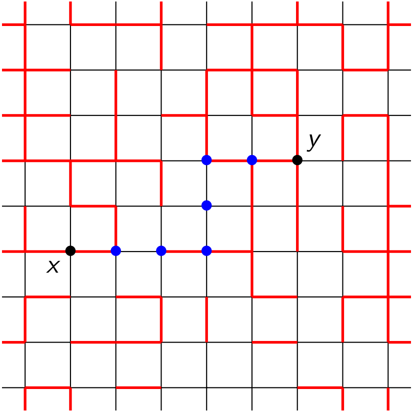

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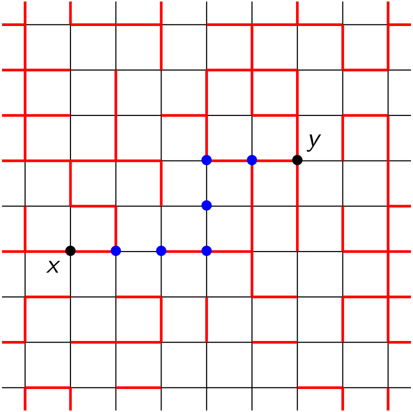

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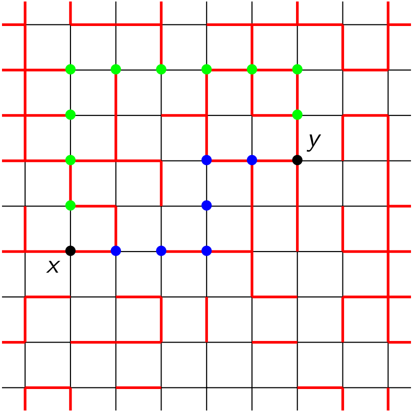

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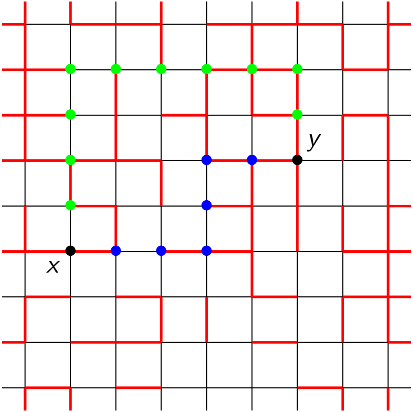

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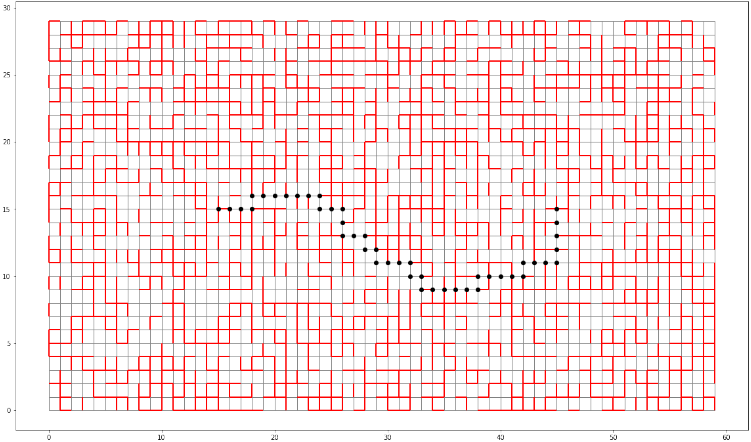

Passage time distribution : $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{10}$. Long edges in red.

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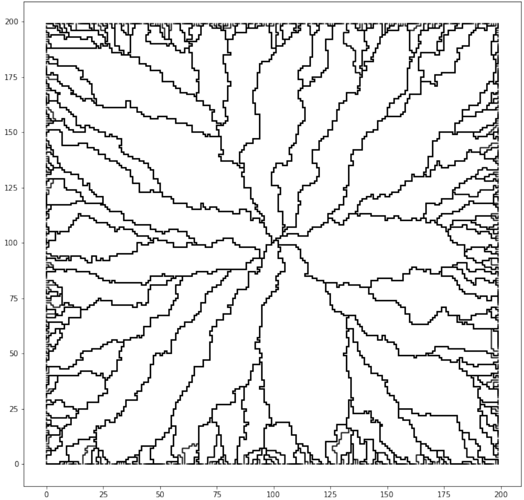

The passage times have exponential distribution with mean 1 .

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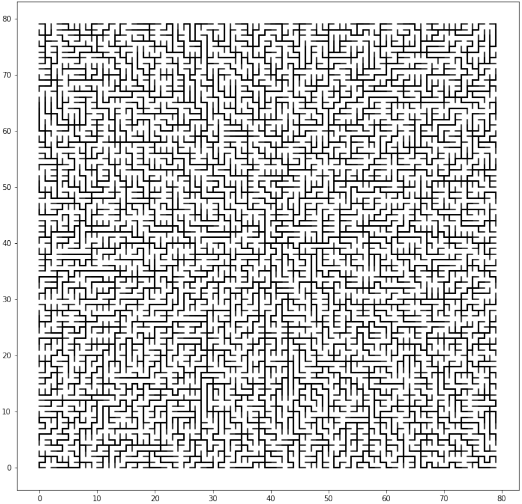

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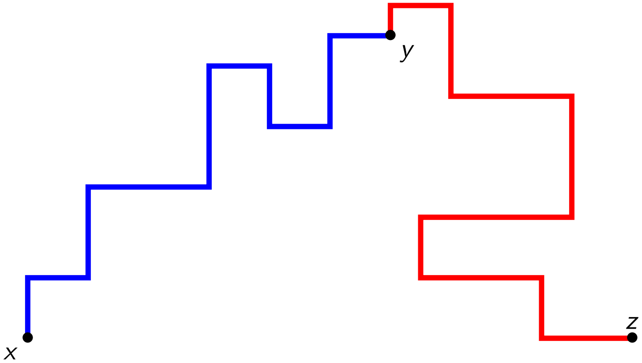
Triangle inequality and random distance



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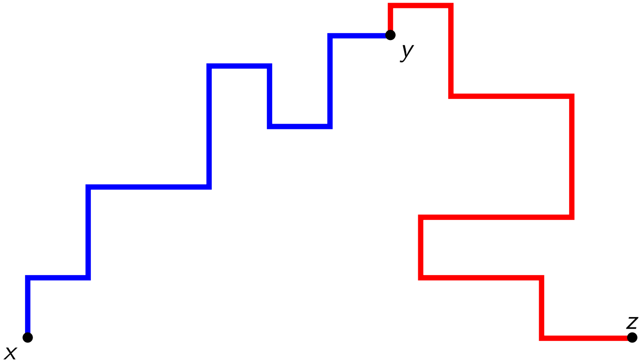

We get $t(x,z) \leq t(x,y) + t(y,z)$.

Objectives of first passage percolation

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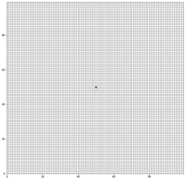
- Understand the properties, particularly asymptotic, of this random distance on $\mathbb{Z}^d$.



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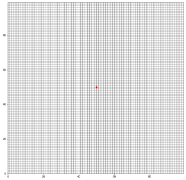
- Describe $B(r) = \{y \in \mathbb{Z}^d \,:\, t(0,y) \le r\}$ for large r.



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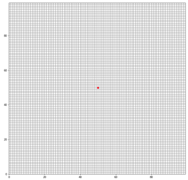
- Understand the properties, particularly asymptotic, of this random distance on $\mathbb{Z}^d$.
- Describe $B(r) = \{y \in \mathbb{Z}^d \,:\, t(0, y) \leq r\}$ for large r.
- Study the geometric properties of geodesics between two points.



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Subadditivity and the time constant

In what follows, we assume that we have good integration assumptions.

Theorem

There exists a constant $\mu(e_1) \in]0, +\infty[$ such that

$$\lim_{n\to\infty} \frac{t(0, ne_1)}{n} = \mu(e_1) \text{ a.s and in } L^1.$$

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Sketch of proof for the following result :

Lemma

There exists $\mu(e_1) \in]0, +\infty[$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[t(0, ne_1)\right]}{n} = \mu(e_1).$$

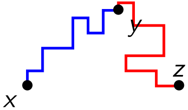
Key = subadditivity.

Subadditivity and the time constant

Lemma

Let $(u_n)_n$ be a sequence such that $u_{n+m} \leq u_n + u_m$ for all positive integers m, n, then the limit of $\left(\frac{u_n}{n}\right)$ exists and we have

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_{n\geq 1} \frac{u_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$



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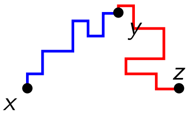

For all m, n positive integers,

$$t(0,(m+n)e_1) \leq t(0,me_1) + t(me_1,(m+n)e_1).$$

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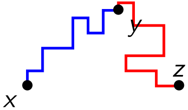

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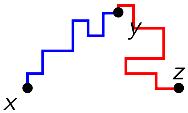

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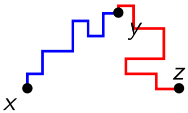

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$$\mathbb{E}\left[t(0,(m+n)e_1)\right] \leq \mathbb{E}\left[t(0,me_1)\right] + \mathbb{E}\left[t(0,ne_1)\right].$$

We take $u_n = \mathbb{E}\left[t(0,ne_1)\right]$ and we can conclude that :

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[t(0,ne_1)\right]}{n} = \inf_{n\geq 1} \frac{\mathbb{E}\left[t(0,ne_1)\right]}{n}.$$

The shape theorem

We can extend the result and obtain for all $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{t(0, \lfloor nx \rfloor)}{n} = \mu(x) \in]0, +\infty[\text{ a.s and in } L^1.$$

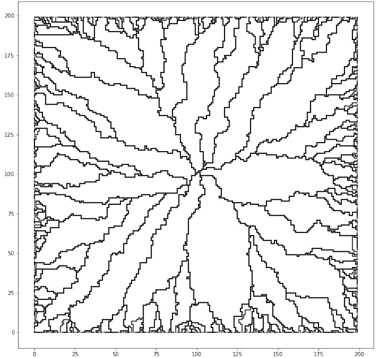
Theorem (Cox - Durett, '83)

We have a deterministic norm μ such that

$$t(x, y) \approx \mu(y - x) \text{ for } \|y - x\| \text{ large.}$$

$$B(r) \approx \text{ball of radius } r \text{ for the norm } \mu.$$

Geodesics



Geodesic between x **and** y **:**
A path γ from x to y such that

$$T(\gamma) = t(x, y).$$

Existence ? Uniqueness ?

Geodesics

Assume that the passage times are not bounded.

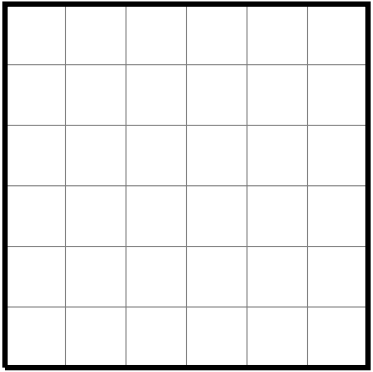
Theorem (Andjel - Vares '12)

For all positive M, there exists $\beta > 0$ and $\alpha > 0$ such that for all $x \in \mathbb{Z}^d$, we have

$$\mathbb{P}\left(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ s.t. } \sum_{e\in\gamma} \mathbb{1}_{\{T(e)>M\}} \leq \alpha\|x\|_1\right) \leq \mathrm{e}^{-\beta\|x\|_1}.$$

Strategy : modification argument.

Patterns



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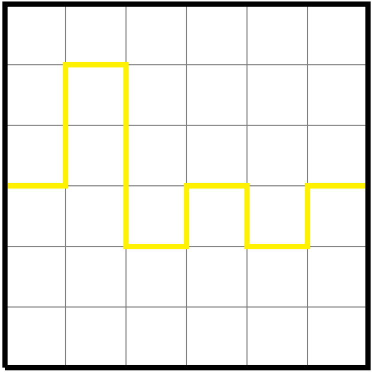

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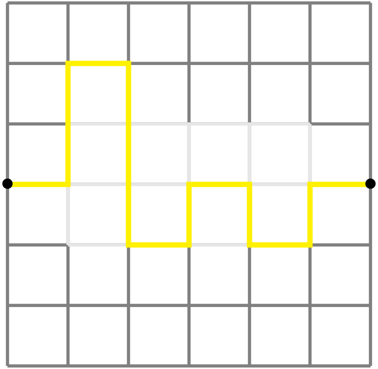

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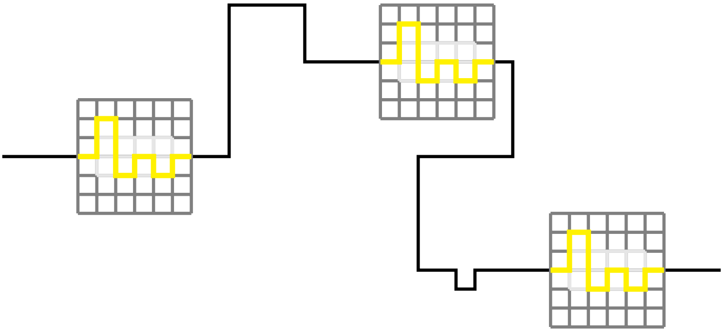

Patterns

Fix a pattern $\mathfrak{P}$.

$\mathcal{N}^{\mathfrak{P}}(\gamma)$: number of patterns $\mathfrak{P}$ visited by the geodesic γ.

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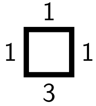
Theorem (J. '23+)

There exist $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that for all $x \in \mathbb{Z}^d$,

$$\mathbb{P}\left(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ s.t. } \mathcal{N}^{\mathfrak{P}}(\gamma) \le \alpha \|x\|_1\right) \le \beta_1 \mathrm{e}^{-\beta_2 \|x\|_1}.$$

Euclidean length of geodesics



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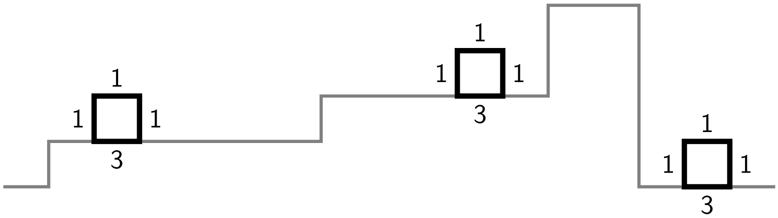

Theorem (Krishnan - Rassoul-Agha - Seppäläinen '21)

There exist $0 < D, \delta, M < +\infty$ s.t.

$$\mathbb{P}\left(\overline{L}_{0,x} - \underline{L}_{0,x} \geq D\|x\|_1\right) \geq \delta \text{ for } \|x\|_1 \geq M.$$

Other applications

Definition

For the probability distributions F and G, we say that F strictly dominates G if $F(x) \leq G(x)$ for all x, but $F \neq G$.

Theorem (Van den Berg - Kesten '93)

Let F and $\tilde{F}$ have finite mean. If F is useful and strictly dominates $\tilde{F}$, then

$$\mu(\tilde{F}) < \mu(F).$$

Theorem (Nakajima '18)

Suppose there exists $\alpha \in (0, +\infty)$ such that $\mathbb{P}(T(e) = \alpha) > 0$. The number of geodesics has an exponential growth.

The end

Thank you !